Comment. Math.
Univ. St. Pauli
XIX-2, 1970

On generalized Meijer function of two variables and some applications

by

Manilal SHAH

(Received August 12, 1970)

Abstract

In the present paper we have evaluated some integrals involving generalized Meijer functions of two variables. These integrals have been employed in solving the partial differential equation of free oscillation of water in a circular lake, and in obtaining Fourier series expansion-formulae for generalized Meijer functions and some integrals with the applications of Mellin's and Laplace's inversion formulae. We have also derived the double-integral-expansion-analogues for generalized Meijer functions of two variables. A number of known, new and interesting results are also given as particular cases of our formulae on specializing the parameters.

1. Introduction

Generalized Meijer function of two variables. Recently, Sharma [13, p. 26-40] has introduced the generalized Meijer function of two variables by means of a double-Mellin-Barnes type contour integral as follows:

(1.1)
$$S\begin{bmatrix} x \\ y \end{bmatrix} \equiv S\left[\begin{array}{c|c|c} \begin{bmatrix} p\ , & 0 \\ A-p, & B \end{bmatrix} & (a);\ (b) & \\ \begin{pmatrix} q\ , & r \\ C-q, & D-r \end{pmatrix} & (c);\ (d) & \begin{matrix} x \\ y \end{matrix} \\ \begin{pmatrix} k\ , & l \\ E-k, & F-l \end{pmatrix} & (e);\ (f) & \end{array}\right]$$
$$=\frac{1}{(2\pi i)^2}\int_{L_1}\int_{L_2} \Phi(s+t)\psi(s, t)x^s y^t ds dt\ ,$$

in which

$$\Phi(s+t)=\frac{\prod_{j=1}^{p} \Gamma[a_j+s+t]}{\prod_{j=p+1}^{A} \Gamma[1-a_j-s-t] \prod_{j=1}^{B} \Gamma[b_j+s+t]}\ ,$$

$$\psi(s, t)=\frac{\prod_{j=1}^{q} \Gamma[1-c_j+s] \prod_{j=1}^{r} \Gamma[d_j-s] \prod_{j=1}^{k} \Gamma[1-e_j+t] \prod_{j=1}^{l} \Gamma[f_j-t]}{\prod_{j=q+1}^{C} \Gamma[c_j-s] \prod_{j=r+1}^{D} \Gamma[1-d_j+s] \prod_{j=k+1}^{E} \Gamma[e_j-t] \prod_{j=l+1}^{F} \Gamma[1-f_j+t]}\ ,$$

where the contour L_1 in the s-plane runs from $-i\infty$ to $+i\infty$, curving so as to ensure, if necessary, that the poles of $\Gamma(d_j-s)$ $(j=1, 2, \cdots, r)$ lie to the right and the poles of $\Gamma(1-c_j+s)$ $(j=1, 2, \cdots, q)$ and $\Gamma(a_j+s+t)$ $(j=1, 2, \cdots, p)$ to the left of the contour. Similarly the contour L_2 in the t-plane consists of the portion of the imaginary axis from $-i\infty$ to $+i\infty$ along with the necessary loops so as to ensure that the poles of $\Gamma(f_j-t)$ $(j=1, 2, \cdots, l)$ lie to the right and the poles of $\Gamma(1-e_j+t)$ $(j=1, 2, \cdots, k)$ and $\Gamma(a_j+s+t)$ $(j=1, 2, \cdots, p)$ to the left of the contour.

The positive integers A, B, C, etc. satisfy the following inequalities: $D\geqslant 1$, $F\geqslant 1$, $A\geqslant 1$, $B\geqslant 1$, $0\leqslant p\leqslant A$, $0\leqslant q\leqslant C$, $0\leqslant k\leqslant E$, $0\leqslant r\leqslant D$, $0\leqslant l\leqslant F$, $A+C\leqslant B+D$ and $A+E\leqslant B+F$.

The point $x=0$, $y=0$ is a singular point of the partial differential equation satisfied by $S\begin{bmatrix} x \\ y \end{bmatrix}$. The behaviour of $S\begin{bmatrix} x \\ y \end{bmatrix}$ in the neighbourhood of $x=0$, $y=0$ is given by Sharma [14, p. 23–24, (36)] as

$$S\begin{bmatrix} x \\ y \end{bmatrix}=O(|x|^{d_{h_1}}|y|^{f_{h_2}}) ,$$

where $h_1=1, 2, \cdots, r$; $h_2=1, 2, \cdots, l$. Similarly the behaviour of the associated function $S_1\begin{bmatrix} x \\ y \end{bmatrix}$ (which corresponds to the case $p=0$) at infinity is given by Sharma [14, p. 27, (40)] as

$$S_1\begin{bmatrix} x \\ y \end{bmatrix}=O(|x|^{c_{j_1}-1}|y|^{e_{j_2}-1}) ,$$

where $j_1=1, 2\cdots, q$; $j_2=1, 2, \cdots, k$.

The double-integral (1.1) converges under the following conditions:-

$$(1.2)\quad \begin{cases} 2(p+q+r)>A+B+C+D,\ |\arg(x)|<[p+q+r-\frac{1}{2}(A+B+C+D)]\pi , \\ 2(p+k+l)>A+B+E+F,\ |\arg(y)|<[p+k+l-\frac{1}{2}(A+B+E+F)]\pi ; \\ \text{If } A+C=B+D,\ A+E=B+F, \text{ then } |(x)|<1,\ |(y)|<1 . \end{cases}$$

For the sake of brevity and ease in writing (a) stands for $a_1, a_2, \cdots, a_A$ and similarly for (b), (c), (d), (e) and (f). Further, wherever it occurs, the following symbols represent the series:

$$\Delta(m, n)=\frac{n}{m}, \frac{n+1}{m}, \cdots, \frac{n+m-1}{m} ;$$

$$\nabla(m, n)=1-\frac{n}{m}, 1-\frac{n+1}{m}, \cdots, 1-\frac{n+m-1}{m}$$

and similarly the notation $\Delta(\delta, a\pm b)$ stands for $\Delta(\delta, a+b)$, $\Delta(\delta, a-b)$ and also for $\Gamma(a\pm b)$.

Double-hypergeometric-function of higher order. The double hypergeometric function of higher order in two variables has been studied by Kampé de Fériet, J. [8]:

$$(1.3)\qquad F\left[\begin{array}{c|c|c} m & (a_m) & \\ l & (b_l);\ (b'_l) & x \\ n & (c_n) & y \\ p & (d_p);\ (d'_p) & \end{array}\right]=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{\prod_{j=1}^{m}(a_j)_{r+s}\prod_{j=1}^{l}\{(b_j)_r(b'_j)_s\}}{r!\,s!\prod_{j=1}^{n}(c_j)_{r+s}\prod_{j=1}^{p}\{(d_j)_r(d'_j)_s\}}x^r y^s\,,$$

where (a_m) denotes the sequence of elements $a_1, \cdots, a_m$ and similarly for (b_l), (b'_l), (c_n), (d_p) and (d'_p). The series for the F-function converges absolutely, for all complex values of x and y if $m+l<n+p+1$. In case $m+l=p+n+1$, it converges absolutely for all complex values of x and y such that [Ragab, 1965, 11] $|x|+|y|<\min(1, 2^{n-m+1})$.

Recently, Srivastava and Saran (Proc. Camb. Phil. Soc. 1968, 435–437) have employed the notation for Kampé de Fériet function in the form

$$(1.4)\qquad F^{m,l}_{n,p}\left[\begin{array}{c} |a|_m:\ |b, b'|_l \\ |c|_n:\ |d, d'|_p \end{array}\middle|\, x, y\right]=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{\prod_{j=1}^{m}(a_j)_{r+s}\prod_{j=1}^{l}\{(b_j)_r(b'_j)_s\}}{r!\,s!\prod_{j=1}^{n}(c_j)_{r+s}\prod_{j=1}^{p}\{(d_j)_r(d'_j)_s\}}x^r y^s\,.$$

The parameters m, l, n, p are known as the characteristic indices and the parameters $n+p$ and $[n+p+1-(m+l)]$ indicate respectively the order and class of the function. As in other hypergeometric functions, none of the denominator parameters is allowed to be a negative integer.

Appell's functions. Functions F reduce to Appell's [1] four functions F_1, F_2, F_3 and F_4 in the following cases:

(i) When $m=l=n=1$, $p=0$, we obtain

$$(1.5)\qquad F^{1,1}_{1,0}\left[\begin{array}{c} a:\ b, b' \\ c:\ \cdots \end{array}\middle|\, x, y\right]=\sum_{r,s=0}^{\infty}\frac{(a)_{r+s}(b)_r(b')_s}{r!\,s!\,(c)_{r+s}}x^r y^s$$
$$=F_1(a\colon b, b'\colon c\colon x, y)\,.$$

(ii) Setting $m=l=p=1$, $n=0$, we have

$$(1.6)\qquad F^{1,1}_{0,1}\left[\begin{array}{c} a\ :b, b' \\ \cdots: d, d' \end{array}\middle|\, x, y\right]=\sum_{r,s=0}^{\infty}\frac{(a)_{r+s}(b)_r(b')_s}{r!\,s!\,(d)_r(d')_s}x^r y^s$$
$$=F_2(a\colon b, b'\colon d, d'\colon x, y)\,.$$

(iii) Substituting $m=p=0$, $l=2$, $n=1$, we get

$$(1.7)\qquad F^{0,2}_{1,0}\left[\begin{array}{c} \cdots: b_1, b_2;\ b'_1, b'_2 \\ c\ :\quad \cdots \end{array}\middle|\, x, y\right]=\sum_{r,s=0}^{\infty}\frac{(b_1)_r(b'_1)_s(b_2)_r(b'_2)_s}{r!\,s!\,(c)_{r+s}}x^r y^s$$
$$=F_3(b_1, b'_1;\ b_2, b'_2\colon c\colon x, y)\,.$$

(iv) With $m=2$, $l=n=0$, $p=1$, we obtain

$$(1.8)\qquad F^{2,0}_{0,1}\left[\begin{array}{c} a_1, a_2:\ \cdots \\ \cdots\ : d, d' \end{array}\middle|\, x, y\right]=\sum_{r,s=0}^{\infty}\frac{(a_1)_{r+s}(a_2)_{r+s}}{r!\,s!\,(d)_r(d')_s}x^r y^s$$

$$=F_4(a_1, a_2: d, d': x, y) .$$

Generalized hypergeometric functions of single variable. Again, when $m=n$, $l=1$, $p=0$ and $y=x$, the F-function reduces to a generalized hypergeometric function of one variable, i.e.

$$F_{p,0}^{p,1}\left[\begin{matrix}|a|_p: b; b' \\ |c|_p: \cdots\end{matrix}\middle| x, x\right] = {}_{p+1}F_p\left[\begin{matrix}a_1, \cdots, a_p, b+b' \\ c_1, \cdots, c_p\end{matrix}; x\right] . \tag{1.9}$$

Integral transforms. The Mellin transform [4, p. 305]:

$$g(s)=\int_0^\infty x^{s-1}f(x)dx = M_s\{f(x)\} \text{ or } M\{f(x); s\} \tag{1.10}$$

and its inversion formula [4, p. 307, (1)]:

$$f(x)=M_x^{-1}\{g(s)\}=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s}g(s)ds . \tag{1.11}$$

The Laplace transform [4, p. 127]:

$$g(s)=\int_0^\infty e^{-sx}f(x)dx = \alpha_s\{f(x)\} \text{ or } \alpha\{f(x): s\} \tag{1.12}$$

and its inversion formula [4, p. 129, (1)]:

$$f(x)=\alpha_x^{-1}\{g(s)\}=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} g(s)e^{-xs}ds . \tag{1.13}$$

The object of this paper is to evaluate certain integrals involving generalized Meijer functions of two variables. These integrals have been employed in obtaining a formal solution of partial differential equation of free oscillation of water in a circular lake, expansion-formulae for generalized Meijer functions of two variables in series of Bessel and circular-functions. We have also derived their double-integral expansion-analogues. Some integrals involving generalized Meijer functions of two variables have been evaluated with the help of Mellin's and Laplace's inversion formulae. We have also exhibited a number of new, known and particular cases of the results with proper choice of parameters in this paper. Therefore, the formulae established in this paper are of general character.

2. Application of generalized Meijer function of two variables and free oscillation of water in a lake

(a) *Integral involving generalized Meijer functions of two variables and Bessel-functions.*

Firstly we evaluate here the following integral

$$I=\int_0^\infty x^\mu J_\gamma(xy) S\left[\begin{matrix}\alpha x^{2\rho} \\ \beta x^{2\rho}\end{matrix}\right] dx \tag{2.1}$$

where ρ is a positive integer.

Substitute the contour integral (1.1) for $S\begin{bmatrix}\alpha x^{2\rho}\\ \beta x^{2\rho}\end{bmatrix}$ on the right hand of (2.1) and invert the order of integration, which can readily be justified by de la Vallée Poussin's theorem [2, p. 504], taking the conditions stated in (1.2) earlier into account, and then interpreting the inner integral with the help of the known formula [5, p. 22, (7)]:

$$\int_0^\infty x^\mu J_\gamma(xy)(xy)^{1/2}dx=2^{\mu+1/2}y^{-\mu-1}\frac{\Gamma(\frac{1}{2}\mu+\frac{1}{2}\gamma+\frac{3}{4})}{\Gamma(\frac{1}{2}\gamma-\frac{1}{2}\mu+\frac{1}{4})},$$

where $-\mathrm{Re}\,(\gamma)-3/2<\mathrm{Re}\,(\mu)<-\frac{1}{2}$ and $y>0$, and using Gauss's multiplication theorem [6, p. 4], we obtain

$$(2.2)\qquad I=(2\rho)^\mu y^{-\mu-1}\frac{1}{(2\pi i)^2}\int_{L_1}\int_{L_2}\frac{\prod_{j=1}^{p}\Gamma(a_j+s+t)\prod_{j=1}^{q}\Gamma(1-c_j+s)}{\prod_{j=p+1}^{A}\Gamma(1-a_j-s-t)\prod_{j=1}^{B}\Gamma(b_j+s+t)}$$
$$\times\frac{\prod_{j=1}^{r}\Gamma(d_j-s)\prod_{j=1}^{k}\Gamma(1-e_j+t)\prod_{j=1}^{l}\Gamma(f_j-t)}{\prod_{j=q+1}^{C}\Gamma(c_j-s)\prod_{j=r+1}^{D}\Gamma(1-d_j+s)\prod_{j=k+1}^{E}\Gamma(e_j-t)}$$
$$\times\frac{\prod_{i=0}^{\rho-1}\Gamma\Big(\frac{\frac{1}{2}\mu+\frac{1}{2}\gamma+\frac{1}{2}+i}{\rho}+s+t\Big)\alpha^s\beta^t}{\prod_{j=l+1}^{F}\Gamma(1-f_j+t)\prod_{i=0}^{\rho-1}\Gamma\Big(\frac{\frac{1}{2}\gamma-\frac{1}{2}\mu+\frac{1}{2}+i}{\rho}-s-t\Big)}\,ds\,dt\,.$$

The contour L_1 is in the s-plane and runs from $-i\infty$ to $+i\infty$ with loops to ensure, if necessary, that the poles of $\Gamma(d_j-s)$ $(j=1,2,\cdots,r)$ lie to the right and the poles of $\Gamma(1-c_j+s)$, $(j=1,2,\cdots,q)$ and $\Gamma(a_j+s+t)$ $(j=1,2,\cdots,p)$, $\Gamma((\frac{1}{2}\mu+\frac{1}{2}\gamma+\frac{1}{2}+i)/\rho+s+t)$, $(i=0,1,2,\cdots,\rho-1)$ to the left of the contour.

Similarly the contour L_2 in the t-plane consists of the portion of the imaginary axis from $-i\infty$ to $+i\infty$ along with the necessary loops so as to ensure that the poles of $\Gamma(f_j-t)$ $(j=1,2,\cdots,l)$ lie to the right and the poles of $\Gamma(1-e_j+t)$ $(j=1,2,\cdots,k)$ and $\Gamma(a_j+s+t)$ $(j=1,2,\cdots,p)$, $\Gamma((\frac{1}{2}\mu+\frac{1}{2}\gamma+\frac{1}{2}+i)/\rho+s+t)$ $(i=0,1,\cdots,\rho-1)$ to the left of the contour.

On interpreting (2.2) by virtue of (1.1), we have

$$(2.3)\qquad I=(2\rho)^\mu y^{-\mu-1}S\left[\begin{array}{c|c|c}\begin{bmatrix}p+\rho\,, & 0\\ A-p+\rho, & B\end{bmatrix} & \begin{array}{r}\Delta(\rho,\frac{1}{2}\mu+\frac{1}{2}\gamma+\frac{1}{2}),(a),\\ \Delta(\rho,\frac{1}{2}\mu-\frac{1}{2}\gamma+\frac{1}{2});\ (b)\end{array} & \alpha\Big(\frac{2\rho}{y}\Big)^{2\rho}\\ \begin{pmatrix}q & , r\\ C-q & ,D-r\end{pmatrix} & (c);(d) & \\ \begin{pmatrix}k & , l\\ E-k & , F-l\end{pmatrix} & (e);(f) & \beta\Big(\frac{2\rho}{y}\Big)^{2\rho}\end{array}\right],$$

provided $-\mathrm{Re}\,(\gamma)-3/2<\mathrm{Re}\,[\mu+2\rho d_{h_1}+2\rho f_{h_2}]<-\frac{1}{2}$ $(h_1=1, 2, \cdots, r;\ h_2=1, 2, \cdots, l)$, $y>0$, and either the condition (1.2) stated earlier holds, or

$$A+C<B+D, \quad A+E<B+F\ ,$$

or else

$$A+C=B+D, \quad A+E=B+F \ \text{ with } \ |(\alpha)|<1, \quad |(\beta)|<1\ .$$

(b) *Expansion formula for generalized Meijer function of two variables in series of Bessel-functions.*

We establish here an expansion-formula for generalized Meijer function involving Bessel-functions.

For $0<x<\infty$, $y>0$, let

$$f(x)\equiv x^{\mu}S\begin{bmatrix}\alpha x^{2\rho}\\ \beta x^{2\rho}\end{bmatrix}=\sum_{\xi=0}^{\infty}A_{\xi}J_{\xi}(yx)\ . \tag{2.4}$$

This expansion is valid since $f(x)$ is continuous and of bounded variation in the open interval $(0, \infty)$. Multiply both sides of (2.4) by $x^{-1}J_{\gamma}(xy)$ and integrate with respect to x over $(0, \infty)$ and change the order of integration and summation on the right, which is permissible due to the absolute convergence of the integral and summation involved during the process, we obtain

$$\int_0^{\infty}x^{\mu-1}J_{\gamma}(xy)S\begin{bmatrix}\alpha x^{2\rho}\\ \beta x^{2\rho}\end{bmatrix}dx=\sum_{\xi=0}^{\infty}A_{\xi}\int_0^{\infty}x^{-1}J_{\gamma}(xy)J_{\xi}(xy)dx\ . \tag{2.5}$$

Using the orthogonality-property for Bessel functions [9, p. 291, (6)]:

$$\int_0^{\infty}x^{-1}J_n(x)J_m(x)dx=\begin{cases}0, & \text{if } \ m\neq n\ ,\\ \dfrac{1}{2n}, & \text{if } \ m=n,\ m+n>-1\end{cases}$$

on the right and (2.3) on the left, we have

$$A_{\gamma}=(2\rho)^{\mu-1}y^{-\mu}(2\gamma)S\left[\begin{matrix}\begin{bmatrix}p+\rho\ , & 0\\ A-p+\rho, & B\end{bmatrix}\\ \begin{pmatrix}q\ , & r\\ C-q\ , & D-r\end{pmatrix}\\ \begin{pmatrix}k\ , & l\\ E-k\ , & F-l\end{pmatrix}\end{matrix}\left|\begin{matrix}\Delta(\rho, \frac{1}{2}\mu+\frac{1}{2}\gamma), (a),\\ \Delta(\rho, \frac{1}{2}\mu-\frac{1}{2}\gamma);\ (b)\\ (c); (d)\\ (e); (f)\end{matrix}\right|\begin{matrix}\alpha\left(\dfrac{2\rho}{y}\right)^{2\rho}\\ \beta\left(\dfrac{2\rho}{y}\right)^{2\rho}\end{matrix}\right]. \tag{2.6}$$

In view of (2.4) and (2.6), we obtain the expansion formula

$$x^{\mu}S\begin{bmatrix}\alpha x^{2\rho}\\ \beta x^{2\rho}\end{bmatrix}=(2\rho)^{\mu-1}y^{-\mu}\sum_{\xi=0}^{\infty}(2\xi) \tag{2.7}$$

$$\times S\left[\begin{array}{c|c|c}\begin{bmatrix} p+\rho\ , & 0 \\ A-p+\rho, & B \end{bmatrix} & \begin{array}{l}\Delta(\rho,\ \frac{1}{2}\mu+\frac{1}{2}\xi),\ (a), \\ \Delta(\rho,\ \frac{1}{2}\mu-\frac{1}{2}\xi);\ (b)\end{array} & \alpha\left(\frac{2\rho}{y}\right)^{2\rho} \\ \begin{pmatrix} q\ , & r \\ C-q\ , & D-r \end{pmatrix} & (c);\ (d) & \\ \begin{pmatrix} k\ , & l \\ E-k\ , & F-l \end{pmatrix} & (e);\ (f) & \beta\left(\frac{2\rho}{y}\right)^{2\rho} \end{array}\right] J_\xi(xy)\ ,$$

where ρ is a positive integer, $0<x<\infty$, $y>0$. The expansion formula is valid under the following set of conditions

$$(\mathrm{i})\quad \begin{cases} 2(p+q+r)>A+B+C+D,\ |\arg(\alpha)|<[p+q+r-\frac{1}{2}(A+B+C+D)]\pi\ , \\ 2(p+k+l)>A+B+E+F,\ |\arg(\beta)|<[p+k+l-\frac{1}{2}(A+B+E+F)]\pi\ ; \\ -\mathrm{Re}\ (\xi)-\frac{1}{2}<\mathrm{Re}\ [\mu+2\rho d_{h_1}+2\rho f_{h_2}]<\frac{1}{2} \\ (h_1=1,\ 2,\ \cdots,\ r;\ h_2=1,\ 2,\ \cdots,\ l)\ , \end{cases}$$

$$(\mathrm{ii})\quad \begin{cases} A+C<B+D, \quad A+E<B+F\ , \\ \text{or else } A+C=B+D, \quad A+E=B+F \text{ with } |(\alpha)|<1,\ |(\beta)|<1\ . \end{cases}$$

(c) *Use of generalized Meijer function and free oscillation of water in a lake.*

Here we shall use a generalized Meijer function in obtaining a formal solution of a partial differential equation of free oscillation of water in a circular lake.

The partial differential equation to be considered is as follows [10, p. 45, (1)]:

$$R^2\frac{\partial^2\mathfrak{F}}{\partial R^2}+R\frac{\partial\mathfrak{F}}{\partial R}+\frac{\partial^2\mathfrak{F}}{\partial\theta^2}+K^2R^2\mathfrak{F}=0\ , \tag{2.8}$$

where $\mathfrak{F}$ is the vertical displacement of water surface from its equilibrium position, $K=\omega/c$, $c=\sqrt{(gd)}$, the velocity of a free wave of small amplitude in a large expanse of water of depth d, and g is the acceleration due to gravity.

We shall further assume that:

(i) the lake is stationary in space,

(ii) in any vibrational mode $\mathfrak{F}$ varies harmonically with time and $\mathfrak{F}$ is small enough for neglecting its square,

(iii) there is no loss of energy.

If we further take into account all physical considerations [10, p. 47, Art. 2.6.1], the complete solution of (2.8) is [10, p. 47, (2)]:

$$\mathfrak{F}(R,\ \theta,\ t)=\sum_{\xi=0}^{\infty} A_\xi J_\xi(KR)\cos(\xi\theta-\alpha_\xi)\cos(\omega_\xi t-\varepsilon_\xi)\ . \tag{2.9}$$

Solution of the problem.

In (2.9), if $\theta=t=0$, let

$$\mathfrak{J}(R,\,0,\,0)=f(R)=R^{\mu}S\begin{bmatrix}\alpha R^{2\rho}\\ \beta R^{2\rho}\end{bmatrix}, \tag{2.10}$$

then

$$f(R)=R^{\mu}S\begin{bmatrix}\alpha R^{2\rho}\\ \beta R^{2\rho}\end{bmatrix}=\sum_{\xi=0}^{\infty}A_{\xi}J_{\xi}(KR)\cos\alpha_{\xi}\cos\varepsilon_{\xi}\,. \tag{2.11}$$

Multiply both sides of (2.11) by $R^{-1}J_{\gamma}(KR)$ and integrate with respect to R over $(0,\,\infty)$, change the order of integration and summation (which is admissible), using orthogonality-property for Bessel-functions etc., we have

$$A_{\gamma}=\frac{(2\rho)^{\mu-1}(K)^{-\mu}(2\gamma)}{\cos\alpha_{\gamma}\cos\varepsilon_{\gamma}} \tag{2.12}$$
$$\times S\left[\begin{array}{c|c|c}\begin{bmatrix}p+\rho\,, & 0\\ A-p+\rho, & B\end{bmatrix} & \begin{array}{r}\Delta(\rho,\,\tfrac{1}{2}\mu+\tfrac{1}{2}\gamma),\,(a),\\ \Delta(\rho,\,\tfrac{1}{2}\mu-\tfrac{1}{2}\gamma);\ (b)\end{array} & \alpha\left(\dfrac{2\rho}{K}\right)^{2\rho}\\ \begin{pmatrix}q & , & r\\ C-q & , & D-r\end{pmatrix} & (c);\,(d) & \\ \begin{pmatrix}k & , & l\\ E-k & , & F-l\end{pmatrix} & (e);\,(f) & \beta\left(\dfrac{2\rho}{K}\right)^{2\rho}\end{array}\right].$$

Now by virtue of (2.12), the solution (2.9) reduces to

$$\mathfrak{J}(R,\,\theta,\,t)=(2\rho)^{\mu-1}(K)^{-\mu}\sum_{\xi=0}^{\infty}\frac{(2\xi)}{\cos\alpha_{\xi}\cos\varepsilon_{\xi}} \tag{2.13}$$
$$\times S\left[\begin{array}{c|c|c}\begin{bmatrix}p+\rho\,, & 0\\ A-p+\rho, & B\end{bmatrix} & \begin{array}{r}\Delta(\rho,\,\tfrac{1}{2}\mu+\tfrac{1}{2}\xi),\,(a),\\ \Delta(\rho,\,\tfrac{1}{2}\mu-\tfrac{1}{2}\xi);\ (b)\end{array} & \alpha\left(\dfrac{2\rho}{K}\right)^{2\rho}\\ \begin{pmatrix}q & , & r\\ C-q & , & D-r\end{pmatrix} & (c);\,(d) & \\ \begin{pmatrix}k & , & l\\ E-k & , & F-l\end{pmatrix} & (e);\,(f) & \beta\left(\dfrac{2\rho}{K}\right)^{2\rho}\end{array}\right]$$
$$\times\cos(\xi\theta-\alpha_{\xi})\cos(\omega_{\xi}t-\varepsilon_{\xi})\,J_{\xi}(RK),$$

where ρ is a positive integer, $K>0$. (2.12) and (2.13) are valid under the following conditions

$$\text{(i)}\quad\begin{cases}2(p+q+r)>A+B+C+D,\ |\arg(\alpha)|<[p+q+r-\tfrac{1}{2}(A+B+C+D)]\pi\,,\\ 2(p+k+l)>A+B+E+F,\ |\arg(\beta)|<[p+k+l-\tfrac{1}{2}(A+B+E+F)]\pi\,;\\ -\mathrm{Re}\,(\xi)-\tfrac{1}{2}<\mathrm{Re}\,[\mu+2\rho d_{h_1}+2\rho f_{h_2}]<\tfrac{1}{2}\\ (h_1=1,\,2,\,\cdots,\,r;\ h_2=1,\,2,\,\cdots,\,l)\end{cases}$$

or

$$\text{(ii)}\quad\begin{cases}A+C<B+D,\quad A+E<B+F,\ \text{or else}\\ A+C=B+D,\ A+E=B+F\ \text{with}\ |(\alpha)|<1,\ |(\beta)|<1\,.\end{cases}$$

3. Particular cases of (2.3), (2.7) and (2.13)

(I) Setting $A=B=p=0$, we have

$$(3.1)\quad \int_0^\infty x^\mu J_\gamma(xy) G_{C,D}^{r,q}\left[\alpha x^{2\rho}\,\middle|\,\begin{matrix}(c)\\(d)\end{matrix}\right] G_{E,F}^{l,k}\left[\beta x^{2\rho}\,\middle|\,\begin{matrix}(e)\\(f)\end{matrix}\right]dx$$

$$=(2\rho)^\mu y^{-\mu-1} S\left[\begin{matrix}\begin{bmatrix}\rho\ , & 0\\ \rho\ , & 0\end{bmatrix}\\ \begin{pmatrix}q\ , & r\\ C-q, & D-r\end{pmatrix}\\ \begin{pmatrix}k\ , & l\\ E-k, & F-l\end{pmatrix}\end{matrix}\,\middle|\,\begin{matrix}\Delta(\rho,\tfrac{1}{2}\mu\pm\tfrac{1}{2}\gamma+\tfrac{1}{2});\cdots\\ \\ (c);(d)\\ \\ (e);(f)\end{matrix}\,\middle|\,\begin{matrix}\alpha\left(\dfrac{2\rho}{y}\right)^{2\rho}\\ \\ \beta\left(\dfrac{2\rho}{y}\right)^{2\rho}\end{matrix}\right],$$

$$(3.2)\quad x^\mu G_{C,D}^{r,q}\left[\alpha x^{2\rho}\,\middle|\,\begin{matrix}(c)\\(d)\end{matrix}\right] G_{E,F}^{l,k}\left[\beta x^{2\rho}\,\middle|\,\begin{matrix}(e)\\(f)\end{matrix}\right]$$

$$=(2\rho)^{\mu-1} y^{-\mu}\sum_{\xi=0}^{\infty}(2\xi) S\left[\begin{matrix}\begin{bmatrix}\rho\ , & 0\\ \rho\ , & 0\end{bmatrix}\\ \begin{pmatrix}q\ , & r\\ C-q, & D-r\end{pmatrix}\\ \begin{pmatrix}k\ ; & l\\ E-k, & F-l\end{pmatrix}\end{matrix}\,\middle|\,\begin{matrix}\Delta(\rho,\tfrac{1}{2}\mu\pm\tfrac{1}{2}\xi);\cdots\\ \\ (c);(d)\\ \\ (e);(f)\end{matrix}\,\middle|\,\begin{matrix}\alpha\left(\dfrac{2\rho}{y}\right)^{2\rho}\\ \\ \beta\left(\dfrac{2\rho}{y}\right)^{2\rho}\end{matrix}\right] J_\xi(yx),$$

$$(3.3)\quad \mathfrak{J}(R,\theta,t)=(2\rho)^{\mu-1}(K)^{-\mu}\sum_{\xi=0}^{\infty}\frac{(2\xi)J_\xi(KR)}{\cos\alpha_\xi\cos\varepsilon_\xi}\cos(\xi\theta-\alpha_\xi)\cos(\omega_\xi t-\varepsilon_\xi)$$

$$S\left[\begin{matrix}\begin{bmatrix}\rho\ , & 0\\ \rho\ , & 0\end{bmatrix}\\ \begin{pmatrix}q\ , & r\\ C-q, & D-r\end{pmatrix}\\ \begin{pmatrix}k\ , & l\\ E-k, & F-l\end{pmatrix}\end{matrix}\,\middle|\,\begin{matrix}\Delta(\rho,\tfrac{1}{2}\mu\pm\tfrac{1}{2}\xi);\cdots\\ \\ (c);(d)\\ \\ (e);(f)\end{matrix}\,\middle|\,\begin{matrix}\alpha\left(\dfrac{2\rho}{K}\right)^{2\rho}\\ \\ \beta\left(\dfrac{2\rho}{K}\right)^{2\rho}\end{matrix}\right],$$

where ρ is a positive integer, $K>0$, $y>0$, G is the Meijer G-function [6, p. 207, (1)] and

$$(\mathrm{i})\quad \begin{cases}2(q+r)>C+D,\ |\arg(\alpha)|<[q+r-\tfrac{1}{2}(C+D)]\pi\ ,\\ 2(k+l)>E+F,\ |\arg(\beta)|<[k+l-\tfrac{1}{2}(E+F)]\pi\ ;\\ -\mathrm{Re}(\gamma)-3/2<\mathrm{Re}[\mu+2\rho d_{h_1}+2\rho f_{h_2}]<-\tfrac{1}{2}\ ;\\ -\mathrm{Re}(\xi)-1/2<\mathrm{Re}[\mu+2\rho d_{h_1}+2\rho f_{h_2}]<\tfrac{1}{2}\\ (h_1=1,2,\cdots,r;\ h_2=1,2,\cdots,l)\end{cases}$$

or

$$(\mathrm{ii})\quad \begin{cases}C<D,\quad E<F\ ,\\ \text{or else } C=D,\quad E=F \text{ with } |(\alpha)|<1,\ |(\beta)|<1\ .\end{cases}$$

(II) Substituting $A=E=p=k=l-1=F-1=0$, replace $A+C$ by A, $B+D$ by B, $A+q$ by s together with necessary changes in the parameters etc., and then let $\beta\to 0$, thus we obtain

$$(3.4)\qquad \int_0^\infty x^\mu J_\gamma(xy) G_{A,B}^{r,s}\left[\alpha x^{2\rho}\,\middle|\,\begin{matrix}(a)\\(b)\end{matrix}\right]dx$$
$$=(2\rho)^\mu y^{-\mu-1} G_{A+2\rho,B}^{r,s+\rho}\left[\alpha\left(\frac{2\rho}{y}\right)^{2\rho}\,\middle|\,\begin{matrix}\nabla(\rho,\tfrac12\mu+\tfrac12\gamma+\tfrac12),(a),\nabla(\rho,\tfrac12-\tfrac12\gamma+\tfrac12\mu)\\(b)\end{matrix}\right],$$

$$(3.5)\qquad x^\mu G_{A,B}^{r,s}\left[\alpha x^{2\rho}\,\middle|\,\begin{matrix}(a)\\(b)\end{matrix}\right]$$
$$=(2\rho)^{\mu-1}y^{-\mu}\sum_{\xi=0}^\infty (2\xi) G_{A+2\rho,B}^{r,s+\rho}\left[\alpha\left(\frac{2\rho}{y}\right)^{2\rho}\,\middle|\,\begin{matrix}\nabla(\rho,\tfrac12\mu+\tfrac12\xi),(a),\nabla(\rho,\tfrac12\mu-\tfrac12\xi)\\(b)\end{matrix}\right]J_\xi(xy),$$

$$(3.6)\qquad \mathfrak{Z}(R,\theta,t)=(2\rho)^{\mu-1}(K)^{-\mu}\sum_{\xi=0}^\infty \frac{(2\xi)J_\xi(RK)}{\cos\alpha_\xi\cos\varepsilon_\xi}\cos(\xi\theta-\alpha_\xi)\cos(\omega_\xi t-\varepsilon_\xi)$$
$$\times G_{A+2\rho,B}^{r,s+\rho}\left[\alpha\left(\frac{2\rho}{K}\right)^{2\rho}\,\middle|\,\begin{matrix}\nabla(\rho,\tfrac12\mu+\tfrac12\xi),(a),\nabla(\rho,\tfrac12\mu-\tfrac12\xi)\\(b)\end{matrix}\right]$$

where ρ is a positive integer, $y>0$, $K>0$ and

$$(\mathrm{i})\quad \begin{cases}2(r+s)>A+B,\ |\arg(\alpha)|<\pi[r+s-\tfrac12(A+B)]\,,\\ -\mathrm{Re}(\gamma)-3/2<\mathrm{Re}[\mu+2\rho b_j]<-\tfrac12\,;\\ -\mathrm{Re}(\xi)-\tfrac12<\mathrm{Re}[\mu+2\rho b_j]<\tfrac12\ (j=1,2,\cdots,r)\end{cases}$$

or

(ii) $A<B$, or else $A=B$ with $|(\alpha)|<1$.

Results (3.4) and (3.6) are given by Bajpai [3, (2.5.2), (10.5.7), pp. 49 and 203].

(III) Taking $p=A=m$, $B=n$, $q=k=C=E=l$, $r=l=1$, $D=F=p+1$, $d_1=f_1=0$, replace b_j, $1-c_j$, $1-d_j$, $1-e_j$ and $1-f_j$ by c_j, b_j, d_j, b'_j and d'_j respectively. Thus we obtain

$$(3.7)\qquad \int_0^\infty x^\mu J_\gamma(xy) F\left[\begin{matrix}m\\l\\n\\p\end{matrix}\,\middle|\,\begin{matrix}(a_m)\\(b_l);(b'_l)\\(c_n)\\(d_p);(d'_p)\end{matrix}\,\middle|\,\begin{matrix}\alpha x^{2\rho}\\ \\ \beta x^{2\rho}\end{matrix}\right]dx=(2\rho)^\mu y^{-\mu-1}\prod_{i=0}^{\rho-1}\Gamma\left(\frac{\tfrac12\mu\pm\tfrac12\gamma+\tfrac12+i}{\rho}\right)$$
$$\times F\left[\begin{matrix}m+2\rho\\l\\n\\p\end{matrix}\,\middle|\,\begin{matrix}\Delta(\rho,\tfrac12\mu\pm\tfrac12\gamma+\tfrac12),(a_m)\\(b_l);(b'_l)\\(c_n)\\(d_p);(d'_p)\end{matrix}\,\middle|\,\begin{matrix}\alpha\left(\frac{2\rho}{y}\right)^{2\rho}\\ \beta\left(\frac{2\rho}{y}\right)^{2\rho}\end{matrix}\right].$$

$$(3.8)\qquad x^\mu F\left[\begin{matrix}m\\l\\n\\p\end{matrix}\,\middle|\,\begin{matrix}(a_m)\\(b_l);(b'_l)\\(c_n)\\(d_p);(d'_p)\end{matrix}\,\middle|\,\begin{matrix}\alpha x^{2\rho}\\ \\ \beta x^{2\rho}\end{matrix}\right]=(2\rho)^{\mu-1}y^{-\mu}\sum_{\xi=0}^\infty(2\xi)\prod_{i=0}^{\rho-1}\Gamma\left(\frac{\tfrac12\mu\pm\tfrac12\xi+i}{\rho}\right)$$

$$\times F\begin{bmatrix} m+2\rho & \Delta(\rho, \tfrac{1}{2}\mu\pm\tfrac{1}{2}\xi), (a_m) & \alpha\left(\dfrac{2\rho}{y}\right)^{2\rho} \\ l & (b_l); (b'_l) & \\ n & (c_n) & \\ p & (d_p); (d'_p) & \beta\left(\dfrac{2\rho}{y}\right)^{2\rho} \end{bmatrix} J_\xi(xy) .$$

(3.9) $$\mathfrak{F}(R, \theta, t) = (2\rho)^{\mu-1}K^{-\mu}\sum_{\xi=0}^{\infty} \frac{(2\xi)J_\xi(KR)}{\cos\alpha_\xi \cos\varepsilon_\xi} \prod_{i=0}^{\rho-1} \Gamma\left(\frac{\frac{1}{2}\mu\pm\frac{1}{2}\xi+i}{\rho}\right)$$

$$\times F\begin{bmatrix} m+2\rho & \Delta(\rho, \tfrac{1}{2}\mu\pm\tfrac{1}{2}\xi), (a_m) & \alpha\left(\dfrac{2\rho}{K}\right)^{2\rho} \\ l & (b_l); (b'_l) & \\ n & (c_n) & \\ p & (d_p); (d'_p) & \beta\left(\dfrac{2\rho}{K}\right)^{2\rho} \end{bmatrix} \cos(\xi\theta-\alpha_\xi)\cos(\omega_\xi t-\varepsilon_\xi) ,$$

where ρ is a positive integer, $y>0$, $K>0$. (3.7), (3.8) and (3.9) are valid if $m+l<n+p+1$, ($m+l=n+p+1$; then $|(\alpha)|<1$, $|(\beta)|<1$),

$$-\mathrm{Re}(\gamma)-3/2<\mathrm{Re}(\mu)<-\tfrac{1}{2} \quad \text{and} \quad -\mathrm{Re}(\xi)-\tfrac{1}{2}<\mathrm{Re}(\mu)<\tfrac{1}{2} ,$$

or

if $m+l+1>n+p$, then

$$|\arg(\beta)|, |\arg(\alpha)|<(m+l-n-p)\pi/2 .$$

4. Results involving generalized hypergeometric functions of single variable and Bessel-functions

We have also seen that the F-functions are further reduced to the generalized hypergeometric functions of one variable by taking $m=n$, $l=1$, $p=0$ and $\beta=\alpha$ etc. Thus we obtain

(4.1) $$\int_0^\infty x^\mu J_\gamma(xy)\,{}_{p+1}F_p\left\{\begin{matrix}(a_p), b+b' \\ (c_p)\end{matrix}; \alpha x^{2\rho}\right\}dx = (2\rho)^\mu y^{-\mu-1}$$

$$\times \prod_{i=0}^{\rho-1}\Gamma\left(\frac{\frac{1}{2}\mu\pm\frac{1}{2}\gamma+\frac{1}{2}+i}{\rho}\right) {}_{p+2\rho+1}F_p\left\{\begin{matrix}\Delta(\rho, \frac{1}{2}\mu\pm\frac{1}{2}\gamma+\frac{1}{2}), (a_p), b+b' \\ (c_p)\end{matrix}; \alpha\left(\frac{2\rho}{y}\right)^{2\rho}\right\} ,$$

(4.2) $$x^\mu\,{}_{p+1}F_p\left\{\begin{matrix}(a_p), b+b' \\ (c_p)\end{matrix}; \alpha x^{2\rho}\right\} = (2\rho)^{\mu-1}y^{-\mu}\sum_{\xi=0}^{\infty}(2\xi)$$

$$\times \prod_{i=0}^{\rho-1}\Gamma\left(\frac{\frac{1}{2}\mu\pm\frac{1}{2}\xi+i}{\rho}\right) {}_{p+2\rho+1}F_p\left\{\begin{matrix}\Delta(\rho, \frac{1}{2}\mu\pm\frac{1}{2}\xi), (a_p), b+b' \\ (c_p)\end{matrix}; \alpha\left(\frac{2\rho}{y}\right)^{2\rho}\right\} J_\xi(xy) ,$$

(4.3) $$\mathfrak{F}(R, \theta, t) = (2\rho)^{\mu-1}(K)^{-\mu}\sum_{\xi=0}^{\infty} \frac{(2\xi)J_\xi(KR)}{\cos\alpha_\xi \cos\varepsilon_\xi} \prod_{i=0}^{\rho-1} \Gamma\left(\frac{\frac{1}{2}\mu\pm\frac{1}{2}\xi+i}{\rho}\right)$$

$$\times {}_{p+2\rho+1}F_p\left\{\begin{matrix}\Delta(\rho, \frac{1}{2}\mu\pm\frac{1}{2}\xi), (a_p), b+b' \\ (c_p)\end{matrix}; \alpha\left(\frac{2\rho}{K}\right)^{2\rho}\right\} \cos(\xi\theta-\alpha_\xi)\cos(\omega_\xi t-\varepsilon_\xi) ,$$

where ρ is a positive integer, $y>0$, $K>0$ and

$$-\mathrm{Re}(\gamma)-3/2<\mathrm{Re}(\mu)<-\tfrac{1}{2} \quad \text{and} \quad -\mathrm{Re}(\xi)-\tfrac{1}{2}<\mathrm{Re}(\mu)<\tfrac{1}{2} .$$

5. Formulae involving Sister Celine's polynomials and Bessel functions

In (4.1), (4.2) and (4.3), adjusting $\rho=\alpha=1$, $b+b'=-n$, $a_1=n+\alpha+\beta+1$, $c_1=1+\alpha$, $c_2=\frac{1}{2}$ and multiplying both sides by $(1+\alpha)_n/n!$, we have

$$\text{(5.1)}\qquad \int_0^\infty x^\mu J_\gamma(xy) f_n^{(\alpha,\beta)}\begin{pmatrix} a_2, \cdots, a_p \\ c_3, \cdots, c_p \end{pmatrix}; x^2 \Big) dx$$

$$=2^\mu y^{-\mu-1}\Gamma(\tfrac{1}{2}\mu\pm\tfrac{1}{2}\gamma+1) f_n^{(\alpha,\beta)}\begin{pmatrix} a_2, \cdots, a_p, \tfrac{1}{2}\mu\pm\tfrac{1}{2}\gamma+\tfrac{1}{2} \\ c_3, \cdots, c_p \end{pmatrix}; \left\{\frac{2}{y}\right\}^2 \Big),$$

$$\text{(5.2)}\qquad x^\mu f_n^{(\alpha,\beta)}\begin{pmatrix} a_2, \cdots, a_p \\ c_3, \cdots, c_p \end{pmatrix}; x^2 \Big)$$

$$=2^{\mu-1} y^{-\mu}\sum_{\xi=0}^\infty (2\xi)\Gamma(\tfrac{1}{2}\mu\pm\tfrac{1}{2}\xi) f_n^{(\alpha,\beta)}\begin{pmatrix} a_2, \cdots, a_p, \tfrac{1}{2}\mu\pm\tfrac{1}{2}\xi \\ c_3, \cdots, c_p \end{pmatrix}; \left\{\frac{2}{y}\right\}^2 \Big) J_\xi(xy),$$

$$\text{(5.3)}\qquad \mathfrak{J}(R,\theta,t)=2^{\mu-1}K^{-\mu}\sum_{\xi=0}^\infty \frac{(2\xi)J_\xi(KR)}{\cos\alpha_\xi\cos\varepsilon_\xi}\Gamma(\tfrac{1}{2}\mu\pm\tfrac{1}{2}\xi)$$

$$\times f_n^{(\alpha,\beta)}\begin{pmatrix} a_2, \cdots, a_p, \tfrac{1}{2}\mu\pm\tfrac{1}{2}\xi \\ c_3, \cdots, c_p \end{pmatrix}; \left\{\frac{2}{K}\right\}^2 \Big)\cos(\xi\theta-\alpha_\xi)\cos(\omega_\xi t-\varepsilon_\xi),$$

where $y>0$, $K>0$, $-\mathrm{Re}(\gamma)-3/2<\mathrm{Re}(\mu)<-1/2$; $-\mathrm{Re}(\xi)-\frac{1}{2}<\mathrm{Re}(\mu)<\frac{1}{2}$, and

$$f_n^{(\alpha,\beta)}\begin{pmatrix} a_2, \cdots, a_p \\ b_3, \cdots, b_q \end{pmatrix}; x \Big)=\frac{(1+\alpha)_n}{n!}\,{}_{p+1}F_q\left[\begin{matrix} -n, n+\alpha+\beta+1, a_2, \cdots, a_p \\ 1+\alpha, \frac{1}{2}, b_3, \cdots, b_q \end{matrix}; x\right]$$

is a generalized Sister Celine's polynomial [15, p. 80, (2.2)], which can be reduced to Sister Celine's polynomial [7, p. 806, (1)] when $\alpha=\beta=0$.

6. Double-integral-expansion-analogues for generalized Meijer functions of two variables associated with Bessel-functions

Adopting the same procedure as in the preceding sections, we can easily prove

(i) *Double-integral-analogues*:

$$\text{(6.1)}\qquad \int_0^\infty\int_0^\infty x^\mu y^\lambda J_\gamma(ux)J_\delta(vy) S\left[\begin{matrix}\alpha x^{2m}\\ \beta y^{2n}\end{matrix}\right]dx\,dy=(2m)^\mu u^{-\mu-1}(2n)^\lambda v^{-\lambda-1}$$

$$\times S\left[\begin{matrix} \begin{bmatrix} p, & 0 \\ A-p, & B\end{bmatrix} \\ \begin{pmatrix} q+m, & r \\ C-q+m, & D-r\end{pmatrix} \\ \begin{pmatrix} k+n, & l \\ E-k+n, & F-l\end{pmatrix}\end{matrix}\left|\begin{matrix} (a);(b) \\ \Delta\left(m, \frac{1-\mu-\gamma}{2}\right), (c) \\ \Delta\left(m, \frac{1-\mu+\gamma}{2}\right); (d) \\ \Delta\left(n, \frac{1-\lambda-\delta}{2}\right), (e) \\ \Delta\left(n, \frac{1-\lambda+\delta}{2}\right); (f)\end{matrix}\right|\begin{matrix} \alpha\left(\frac{2m}{u}\right)^{2m} \\ \beta\left(\frac{2n}{v}\right)^{2n}\end{matrix}\right].$$

(ii) *Double-expansion-analogues:*

$$(6.2)\qquad x^{\mu}y^{\lambda}S\left[\begin{matrix}\alpha x^{2m}\\ \beta y^{2n}\end{matrix}\right]=(2m)^{\mu-1}u^{-\mu}(2n)^{\lambda-1}v^{-\lambda}\sum_{\xi=0}^{\infty}\sum_{\eta=0}^{\infty}(2\xi)(2\eta)$$

$$\times S\left[\begin{matrix}\left[\begin{matrix}p & , & 0\\ A-p & , & B\end{matrix}\right]\\ \left(\begin{matrix}q+m & , & r\\ C-q+m, & D-r\end{matrix}\right)\\ \left(\begin{matrix}k+n & , & l\\ E-k+n, & F-l\end{matrix}\right)\end{matrix}\left|\begin{matrix}(a);(b)\\ \Delta\left(m,\ \dfrac{2-\mu-\xi}{2}\right),\ (c),\ \Delta\left(m,\ \dfrac{2-\mu+\xi}{2}\right);\ (d)\\ \Delta\left(n,\ \dfrac{2-\lambda-\eta}{2}\right),\ (e),\ \Delta\left(n,\ \dfrac{2-\lambda+\eta}{2}\right);\ (f)\end{matrix}\right|\ \begin{matrix}\alpha\left(\dfrac{2m}{u}\right)^{2m}\\ \beta\left(\dfrac{2n}{v}\right)^{2n}\end{matrix}\right]J_{\xi}(ux)J_{\eta}(vy)\ ,$$

where m and n are positive integers, $u>0$, $v>0$. One set of conditions of validity of these formulae include (1.2) and

$$(\mathrm{i})\quad\begin{cases}-\mathrm{Re}\,(\gamma)-3/2<\mathrm{Re}\,[\mu+2md_{h_1}]<-\frac{1}{2}\ ,\\ -\mathrm{Re}\,(\delta)-3/2<\mathrm{Re}\,[\lambda+2nf_{h_2}]<-\frac{1}{2}\ ,\\ -\mathrm{Re}\,(\xi)-1/2<\mathrm{Re}\,[\mu+2md_{h_1}]<\frac{1}{2}\ ,\\ -\mathrm{Re}\,(\eta)-1/2<\mathrm{Re}\,[\lambda+2nf_{h_2}]<\frac{1}{2}\\ (h_1=1,2,\cdots,r;\ h_2=1,2,\cdots,l)\end{cases}$$

or

$$(\mathrm{ii})\quad\begin{cases}A+C<B+D,\quad A+E<B+F\ ,\\ \text{or else}\quad A+C=B+D,\quad A+E=B+F\quad\text{with } |(\alpha)|<1\ ,\\ |(\beta)|<1\ .\end{cases}$$

These results are similar to our earlier results (2.3) and (2.7).

7. Fourier series for generalized Meijer functions

In this section we have established two Fourier series for generalized Meijer functions $S\left[\begin{matrix}x\begin{Bmatrix}\sin\theta\\ \cos\theta\end{Bmatrix}^{2\delta}\\ y\begin{Bmatrix}\sin\theta\\ \cos\theta\end{Bmatrix}^{2\delta}\end{matrix}\right]$ of two variables:

First Fourier series:

$$(7.1)\qquad (\sin\theta)^v S\begin{bmatrix} x(\sin\theta)^{2\delta} \\ y(\sin\theta)^{2\delta}\end{bmatrix} = \frac{2}{\pi^{3/2}\sqrt{\delta}}\sum_{\xi=0}^{\infty}\Gamma(\tfrac{1}{2}\pm\xi)$$

$$\times S\left[\begin{matrix}\begin{bmatrix} p+2\delta, & 0 \\ A-p, & B+2\delta\end{bmatrix} \\ \begin{pmatrix} q, & r \\ C-q, & D-r\end{pmatrix} \\ \begin{pmatrix} k, & l \\ E-k, & F-l\end{pmatrix}\end{matrix}\left|\begin{matrix}\Delta\left(\delta, \frac{v+1}{2}\right), \Delta\left(\delta, \frac{v+2}{2}\right), (a); \\ \Delta(\delta, \tfrac{1}{2}v\pm\xi+1), (b) \\ (c); (d) \\ (e); (f)\end{matrix}\right|\begin{matrix} x \\ \\ y\end{matrix}\right]\cos 2\xi\theta\ .$$

Second Fourier series:

$$(7.2)\qquad (\cos\theta)^v S\begin{bmatrix} x(\cos\theta)^{2\delta} \\ y(\cos\theta)^{2\delta}\end{bmatrix} = \frac{2}{\delta^{1/2}\sqrt{(\pi)}}$$

$$\times \sum_{\xi=0}^{\infty} S\left[\begin{matrix}\begin{bmatrix} p+2\delta, & 0 \\ A-p, & B+2\delta\end{bmatrix} \\ \begin{pmatrix} q, & r \\ C-q, & D-r\end{pmatrix} \\ \begin{pmatrix} k, & l \\ E-k, & F-l\end{pmatrix}\end{matrix}\left|\begin{matrix}\Delta\left(\delta, \frac{v+1}{2}\right), \Delta\left(\delta, \frac{v+2}{2}\right), (a); \\ \Delta(\delta, \tfrac{1}{2}v\pm\tfrac{1}{2}\xi+1), (b) \\ (c); (d) \\ (e); (f)\end{matrix}\right|\begin{matrix} x \\ \\ y\end{matrix}\right]\cos \xi\theta\ ,$$

where δ is a positive integer, $0<\theta<\pi/2$. The above expansion formulae are valid under the following sets of conditions

$$(\mathrm{i})\quad \begin{cases} A+B+C+D<2(p+q+r), \\ \qquad |\arg(x)|<[p+q+r-\tfrac{1}{2}(A+B+C+D)]\pi\ , \\ A+B+E+F<2(p+k+l), \\ \qquad |\arg(y)|<[p+k+l-\tfrac{1}{2}(A+B+E+F)]\pi\ ; \\ \operatorname{Re}(v+2\delta d_{h_1}+2\delta f_{h_2})>0 \quad (h_1=1, 2, \cdots, r;\ h_2=1, 2, \cdots, l)\end{cases}$$

or

$$(\mathrm{ii})\quad \begin{cases} A+C<B+D,\ A+E<B+F\ (A+C=B+D,\ A+E=B+F\ , \\ \text{then } |x|, |y|<1),\ \operatorname{Re}(v+2\delta d_{h_1}+2\delta f_{h_2})>0 \\ (h_1=1, 2, \cdots, r;\ h_2=1, 2, \cdots, l)\ .\end{cases}$$

Proof. (a) For $0<\theta<\pi/2$, let

$$(7.3)\qquad f(\theta)\equiv(\sin\theta)^v S\begin{bmatrix} x(\sin\theta)^{2\delta} \\ y(\sin\theta)^{2\delta}\end{bmatrix} = \sum_{\xi=0}^{\infty} P_\xi\cos 2\xi\theta\ ,$$

where δ is a positive integer.

Equation (7.3) is valid since $f(\theta)$ is continuous and of bounded variation in the open interval $(0, \pi/2)$. Now multiply both sides of (7.3) by $\cos 2u\theta$ and integrate with respect to θ from 0 to $\pi/2$ and change the order of integration and summation (which is permitted) on the right, we obtain

$$(7.4)\qquad \int_0^{\pi/2} \cos 2u\theta (\sin\theta)^v S\begin{bmatrix} x(\sin\theta)^{2\delta} \\ y(\sin\theta)^{2\delta}\end{bmatrix} d\theta = \sum_{\xi=0}^{\infty} P_\xi \int_0^{\pi/2} \cos 2u\theta \cos 2\xi\theta \, d\theta\ .$$

Now using orthogonality-property:

$$\int_{-c}^{c} \cos\left(\frac{n\pi x}{c}\right)\cos\left(\frac{m\pi x}{c}\right) dx = \begin{cases} 0 & \text{for } n\neq m\ , \\ c & \text{for } n=m \end{cases}$$

for cosine functions on the right and evaluating the integral on the left of (7.4) with the help of the following integral:

$$(7.5)\qquad \int_0^{\pi/2} \cos 2u\theta\ (\sin\theta)^v S\begin{bmatrix} x(\sin\theta)^{2\delta} \\ y(\sin\theta)^{2\delta}\end{bmatrix} d\theta = \frac{\Gamma(\frac{1}{2}\pm u)}{2\sqrt{(\pi)}\delta^{1/2}}$$

$$\times S\left[\begin{array}{c|c|c} \begin{bmatrix} p+2\delta, & 0 \\ A-p, & B+2\delta \end{bmatrix} & \Delta\left(\delta, \frac{v+1}{2}\right), \Delta\left(\delta, \frac{v+2}{2}\right), (a); & \\ & \Delta(\delta, \frac{1}{2}v\pm u+1), (b) & x \\ \begin{pmatrix} q & , & r \\ C-q, & D-r \end{pmatrix} & (c); (d) & \\ \begin{pmatrix} k & , & l \\ E-k, & F-l \end{pmatrix} & & y \\ & (e); (f) & \end{array}\right],$$

where δ is a positive integer, $0<\theta<\pi/2$, $u=0, 1, 2, \cdots$, and

$$(\mathrm{i})\quad \begin{cases} A+B+C+D<2(p+q+r), \\ \qquad\qquad |\arg(x)|<[p+q+r-\frac{1}{2}(A+B+C+D)]\pi\ , \\ A+B+E+F<2(p+k+l), \\ \qquad\qquad |\arg(y)|<[p+k+l-\frac{1}{2}(A+B+E+F)]\pi\ , \\ \mathrm{Re}\,(v+2\delta d_{h_1}+2\delta f_{h_2})>0\ \ (h_1=1, 2, \cdots, r;\ h_2=1, 2, \cdots, l) \end{cases}$$

or

$$(\mathrm{ii})\quad \begin{cases} A+C<B+D, \quad A+E<B+F \quad \text{or else} \\ A+C=B+D, \quad A+E=B+F \quad \text{with} \quad |x|<1,\ |y|<1\ , \\ \mathrm{Re}\,(v+2\delta d_{h_1}+2\delta f_{h_2})>0\ \ (h_1=1, 2, \cdots, r;\ h_2=1, 2, \cdots, l)\ , \end{cases}$$

we obtain

$$(7.6)\quad P_u = \frac{2\Gamma(\frac{1}{2}\pm u)}{\pi^{3/2}\delta^{1/2}} S\left[\begin{array}{c|c|c} \begin{bmatrix} p+2\delta, & 0 \\ A-p, & B+2\delta \end{bmatrix} & \Delta\left(\delta, \frac{v+1}{2}\right), \Delta\left(\delta, \frac{v+2}{2}\right), (a); & \\ & \Delta(\delta, \frac{1}{2}v\pm u+1), (b) & x \\ \begin{pmatrix} q & , & r \\ C-q, & D-r \end{pmatrix} & (c); (d) & \\ \begin{pmatrix} k & , & l \\ E-k, & F-l \end{pmatrix} & & y \\ & (e); (f) & \end{array}\right].$$

Thus, in view of (7.3) and (7.6), Fourier series (7.1) is obtained.

{*Proof for the integral* (7.5):

Substitute $S\begin{bmatrix} x(\sin\theta)^{2\delta} \\ y(\sin\theta)^{2\delta} \end{bmatrix}$ on the left hand of (7.5) by the value obtained from (1.1), interchange the order of integrations, then we obtain

$$\frac{1}{(2\pi i)^2}\int_{L_1}\int_{L_2}\Phi(s+t)\psi(s,t)\left\{\int_0^{\pi/2}\cos 2u\theta\,(\sin\theta)^{v+2\delta s+2\delta t}d\theta\right\}x^s y^t ds dt\ .$$

Evaluating θ-integral by virtue of the known result [12, p. 41, ex. (9)]:

$$\int_0^{\pi/2}\cos 2u\theta\,(\sin\theta)^v d\theta=\frac{\Gamma(v+1)\Gamma(\frac{1}{2}\pm u)}{2^{v+1}\Gamma(\frac{1}{2}v\pm u+1)},\quad u=0,1,2,\cdots;\ \mathrm{Re}\,(v)>0$$

and making use of the following relations

$$(\alpha)_{2n}=2^{2n}\left(\frac{\alpha}{2}\right)_n\left(\frac{\alpha+1}{2}\right)_n,\quad (\alpha)_n=\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}\ ,$$

and Gauss's multiplication formula [6, p. 4, (11)]:

$$\prod_{k=0}^{m-1}\Gamma\left(z+\frac{k}{m}\right)=(2\pi)^{(1/2)m-1/2}m^{1/2-mz}\Gamma(mz),\quad m=2,3,4,\cdots,$$

we have

$$\frac{\Gamma(\frac{1}{2}\pm u)}{2\delta^{1/2}\sqrt{(\pi)}}\frac{1}{(2\pi i)^2}\int_{L_1}\int_{L_2}\Phi(s+t)\psi(s,t)$$
$$\times\frac{\prod_{i=0}^{\delta-1}\Gamma\left(\frac{v+1+2i}{2\delta}+s+t\right)\prod_{i=0}^{\delta-1}\Gamma\left(\frac{v+2+2i}{2\delta}+s+t\right)}{\prod_{i=0}^{\delta-1}\Gamma\left(\frac{v\pm 2u+2+2i}{2\delta}+s+t\right)}x^s y^t ds dt\ , \tag{7.7}$$

where the contour L_1 is in the s-plane and runs from $-i\infty$ to $+i\infty$ with loops to ensure, if necessary, that the poles of $\Gamma(d_j-s)$ $(j=1,2,\cdots,r)$ lie to the right and the poles of $\Gamma(1-c_j+s)$ $(j=1,2,\cdots,q)$ and

$$\Gamma(a_j+s+t)\ (j=1,2,\cdots,p),\ \Gamma\left(\frac{v+1+2i}{2\delta}+s+t\right),\ \Gamma\left(\frac{v+2+2i}{2\delta}+s+t\right)$$
$$(i=0,1,\cdots,\delta-1)$$

to the left of the contour.

Similarly the contour L_2 is in the t-plane and runs from $-i\infty$ to $+i\infty$ with loops to ensure, if necessary, that the poles of $\Gamma(f_j-t)$ $(j=1,2,\cdots,l)$ lie to the right and the poles of $\Gamma(1-e_j+t)$ $(j=1,2,\cdots,k)$ and $\Gamma(a_j+s+t)$ $(j=1,2,\cdots,p)$, $\Gamma((v+1+2i)/2\delta+s+t)$, $\Gamma((v+2+2i)/2\delta+s+t)$ $(i=0,1,\cdots,\delta-1)$ lie to the left of the contour.

Therefore, in accordance with the definition (1.1) of the generalized Meijer function $S\begin{bmatrix} x \\ y \end{bmatrix}$, (7.7) yields the value of the integral (7.5).

Note: Regarding interchange of the order of integration, it is observed that θ-integral is absolutely convergent if $\operatorname{Re}(v+2\delta s+2\delta t)>0$, $u=0, 1, 2, \cdots$; $0<\theta<\pi/2$, the double contour-integral converges absolutely under the conditions referred to earlier, and the convergence of the repeated integral follows from that of the integral in (7.5). Hence the interchange of the order of integration is justified.}

(b) To establish (7.2), let

$$f(\theta)\equiv(\cos\theta)^v S\begin{bmatrix} x(\cos\theta)^{2\delta} \\ y(\cos\theta)^{2\delta}\end{bmatrix}=\sum_{\xi=0}^{\infty} P_\xi \cos\xi\theta\,, \tag{7.8}$$

where $0<\theta<\pi/2$ and δ is a positive integer.

Proceeding on the same lines as above, we multiply both sides of (7.8) by $\cos u\theta$, integrate with respect to θ over $(0, \pi/2)$, change the order of integration and summation, which is easily seen to be justified, then use the orthogonality-property for cosine functions on the right and evaluate the integral on the left with the help of the following formula:

$$\int_0^{\pi/2}\cos u\theta\,(\cos\theta)^v S\begin{Bmatrix} x(\cos\theta)^{2\delta} \\ y(\cos\theta)^{2\delta}\end{Bmatrix}d\theta \tag{7.9}$$

$$=\frac{\sqrt{\pi}}{2\delta^{1/2}}S\left[\begin{array}{c|c|c} \begin{bmatrix} p+2\delta, & 0 \\ A-p, & B+2\delta \end{bmatrix} & \Delta\left(\delta, \frac{v+1}{2}\right), \Delta\left(\delta, \frac{v+2}{2}\right), (a); & \\ & \Delta(\delta, \tfrac{1}{2}v\pm\tfrac{1}{2}u+1), (b) & x \\ \begin{pmatrix} q & , & r \\ C-q, & D-r \end{pmatrix} & (c); (d) & \\ & & y \\ \begin{pmatrix} k & , & l \\ E-k, & F-l \end{pmatrix} & (e); (f) & \end{array}\right],$$

where δ is a positive integer, $u=0, 1, 2, \cdots$, $0<\theta<\pi/2$. The formula is valid under the conditions referred to earlier (7.5), we obtain

$$P_u=\frac{2}{\sqrt{(\pi)}\delta^{1/2}} \tag{7.10}$$

$$\times S\left[\begin{array}{c|c|c} \begin{bmatrix} p+2\delta, & 0 \\ A-p, & B+2\delta \end{bmatrix} & \Delta\left(\delta, \frac{v+1}{2}\right), \Delta\left(\delta, \frac{v+2}{2}\right), (a); & \\ & \Delta(\delta, \tfrac{1}{2}v\pm\tfrac{1}{2}u+1), (b) & x \\ \begin{pmatrix} q & , & r \\ C-q, & D-r \end{pmatrix} & (c); (d) & \\ & & y \\ \begin{pmatrix} k & , & l \\ E-k, & F-l \end{pmatrix} & (e); (f) & \end{array}\right].$$

By virtue of (7.8) and (7.10), we obtain the Fourier series (7.2).

{The integral (7.9) may be evaluated on applying the same procedure as above to (7.5) and using the known integral [12, p. 41, ex. (9)]:

$$\int_0^{\pi/2}\cos m\theta\cos^n\theta d\theta=\frac{\pi\Gamma(n+1)}{2^{n+1}\Gamma(\frac{1}{2}n\pm\frac{1}{2}m+1)},$$

where m is a positive integer, etc.}

8. Particular cases of (7.1) and (7.2)

(a) Substituting $A=B=0$, we obtain the Fourier series for the product of two Meijer's G-functions

$$
(8.1)\qquad (\sin\theta)^v G^{r,q}_{C,D}\left(x\{\sin\theta\}^{2\delta}\,\middle|\,\begin{matrix}(c)\\(d)\end{matrix}\right) G^{l,k}_{E,F}\left(y\{\sin\theta\}^{2\delta}\,\middle|\,\begin{matrix}(e)\\(f)\end{matrix}\right) = \frac{2}{\pi^{3/2}\delta^{1/2}}\sum_{\xi=0}^{\infty}\Gamma(\tfrac12\pm\xi)
$$

$$
\times S\left[\begin{array}{c|r|c} \begin{bmatrix}2\delta\ , & 0\\ 0\ , & 2\delta\end{bmatrix} & \Delta\left(\delta,\dfrac{v+1}{2}\right),\Delta\left(\delta,\dfrac{v+2}{2}\right); & \\ & \Delta(\delta,\tfrac12 v\pm\xi+1) & x\\ \begin{pmatrix} q\ , & r\\ C-q, & D-r\end{pmatrix} & (c);(d) & \\ & & y \\ \begin{pmatrix}k\ , & l\\ E-k, & F-l\end{pmatrix} & (e);(f) & \end{array}\right]\cos 2\xi\theta\ ,
$$

$$
(8.2)\qquad (\cos\theta)^v G^{r,q}_{C,D}\left(x\{\cos\theta\}^{2\delta}\,\middle|\,\begin{matrix}(c)\\(d)\end{matrix}\right) G^{l,k}_{E,F}\left(y\{\cos\theta\}^{2\delta}\,\middle|\,\begin{matrix}(e)\\(f)\end{matrix}\right)
$$

$$
=\frac{2}{\delta^{1/2}\sqrt{(\pi)}}\sum_{\xi=0}^{\infty} S\left[\begin{array}{c|r|c} \begin{bmatrix}2\delta\ , & 0\\ 0\ , & 2\delta\end{bmatrix} & \Delta\left(\delta,\dfrac{v+1}{2}\right),\Delta\left(\delta,\dfrac{v+2}{2}\right); & \\ & \Delta(\delta,\tfrac12 v\pm\tfrac12\xi+1) & x\\ \begin{pmatrix} q\ , & r\\ C-q, & D-r\end{pmatrix} & (c);(d) & \\ & & y \\ \begin{pmatrix}k\ , & l\\ E-k, & F-l\end{pmatrix} & (e);(f) & \end{array}\right]\cos \xi\theta\ ,
$$

where δ is a positive integer, $u=0, 1, 2, \cdots,\ \ 0<\theta<\pi/2$. These formulae are valid under the following conditions

$$
\text{(i)}\quad \begin{cases} 2(q+r)>C+D,\ |\arg(x)|<[q+r-\frac12(C+D)]\pi\ , \\ 2(l+k)>E+F,\ |\arg(y)|<[l+k-\frac12(E+F)]\pi\ ; \\ \mathrm{Re}\,(v+2\delta d_{h_1}+2\delta f_{h_2})>0\ \ (h_1=1, 2, \cdots, r;\ h_2=1, 2, \cdots, l) \end{cases}
$$

or

$$
\text{(ii)}\quad \begin{cases} C<D,\ E<F\ \ (\text{if } C=D,\ E=F, \text{ we must have } |x|, |y|<1)\ , \\ \mathrm{Re}\,(v+2\delta d_{h_1}+2\delta f_{h_2})>0\ \ (h_1=1, 2, \cdots, r;\ h_2=1, 2, \cdots, l)\ . \end{cases}
$$

(b) Setting $A=p$, $E=k$, $l=1$, $f_1=0$, replace $A+C$ by A, $B+D$ by B, $A+q$ by s together with appropriate changes in the parameters and then make $y\to 0$ etc., we have

$$
(8.3)\qquad (\sin\theta)^v G^{r,s}_{A,B}\left(x\{\sin\theta\}^{2\delta}\,\middle|\,\begin{matrix}(a)\\(b)\end{matrix}\right)
$$

$$
=\frac{2}{\delta^{1/2}\pi^{3/2}}\sum_{\xi=0}^{\infty}\Gamma(\tfrac12\pm\xi)G^{r,s+2\delta}_{A+2\delta,B+2\delta}\left(x\,\middle|\,\begin{matrix}\nabla\left(\delta,\dfrac{v+1}{2}\right),\nabla\left(\delta,\dfrac{v+2}{2}\right),(a)\\ (b),\nabla(\delta,\frac12 v\pm\xi+1)\end{matrix}\right)\cos 2\xi\theta\ ,
$$

(8.4)
$$(\cos\theta)^v G_{A,B}^{r,s}\left(x\{\cos\theta\}^{2\delta}\,\middle|\,\begin{matrix}(a)\\(b)\end{matrix}\right)$$
$$=\frac{2}{\delta^{1/2}\sqrt{(\pi)}}\sum_{\xi=0}^{\infty} G_{A+2\delta,B+2\delta}^{r,s+2\delta}\left(x\,\middle|\,\begin{matrix}\nabla\left(\delta,\frac{v+1}{2}\right),\nabla\left(\delta,\frac{v+2}{2}\right),(a)\\(b),\nabla(\delta,\frac{1}{2}v\pm\frac{1}{2}\xi+1)\end{matrix}\right)\cos\xi\theta\,.$$

They are valid for a positive integer δ, $u=0,1,2,\cdots;0<\theta<\pi/2$ and

(i) $\begin{cases}2(r+s)>A+B,\ |\arg(x)|<(r+s-\frac{1}{2}A-\frac{1}{2}B)\pi\,,\\ \mathrm{Re}\,(v+2\delta b_h)>0\ \ (h=1,2,\cdots,r)\,,\end{cases}$

(ii) $A<B$ (or $A=B$, $|x|<1$), $\mathrm{Re}\,(v+2\delta b_h)>0$, $h=1,2,\cdots,r$.

(c) Taking $p=A=m$, $B=n$, $q=k=C=E=l$, $r=l=1$, $D=F=p+1$, $d_1=f_1=0$, replace b_j, $1-c_j$, $1-d_j$, $1-e_j$ and $1-f_j$ by c_j, b_j, d_j, b'_j and d'_j respectively, then $S\begin{bmatrix}x\\y\end{bmatrix}$ reduces to Kampé de Fériet's double hypergeometric function (1.3). Thus we have the Fourier series for the double hypergeometric function

(8.5)
$$(\sin\theta)^v F\left[\begin{array}{c|c|c}m&(a_m)&x(\sin\theta)^{2\delta}\\l&(b_l);(b'_l)&\\n&(c_n)&y(\sin\theta)^{2\delta}\\p&(d_p);(d'_p)&\end{array}\right]$$
$$=\frac{2\prod_{i=0}^{\delta-1}\Gamma\left(\frac{v+1+2i}{2\delta}\right)\prod_{i=0}^{\delta-1}\Gamma\left(\frac{v+2+2i}{2\delta}\right)}{(\pi)^{3/2}\delta^{1/2}}\sum_{\xi=0}^{\infty}\frac{\Gamma(\frac{1}{2}\pm\xi)}{\prod_{i=0}^{\delta-1}\Gamma\left(\frac{v\pm2\xi+2+2i}{2\delta}\right)}$$
$$\times F\left[\begin{array}{c|c|c}m+2\delta&(a_m),\Delta\left(\delta,\frac{v+1}{2}\right),\Delta\left(\delta,\frac{v+2}{2}\right)&x\\l&(b_l);(b'_l)&\\n+2\delta&(c_n),\Delta(\delta,\frac{1}{2}v\pm\xi+1)&y\\p&(d_p);(d'_p)&\end{array}\right]\cos2\xi\theta\,,$$

(8.6)
$$(\cos\theta)^v F\left[\begin{array}{c|c|c}m&(a_m)&x(\cos\theta)^{2\delta}\\l&(b_l);(b'_l)&\\n&(c_n)&y(\cos\theta)^{2\delta}\\p&(d_p);(d'_p)&\end{array}\right]$$
$$=\frac{2\prod_{i=0}^{\delta-1}\Gamma\left(\frac{v+1+2i}{2\delta}\right)\prod_{i=0}^{\delta-1}\Gamma\left(\frac{v+2+2i}{2\delta}\right)}{\delta^{1/2}\sqrt{(\pi)}}\sum_{\xi=0}^{\infty}\frac{1}{\prod_{i=0}^{\delta-1}\Gamma\left(\frac{v\pm\xi+2+2i}{2\delta}\right)}$$

$$\times F\left[\begin{array}{c|c|c} m+2\delta & (a_m),\ \Delta\left(\delta, \dfrac{v+1}{2}\right),\ \Delta\left(\delta, \dfrac{v+2}{2}\right) & x \\ l & (b_l);\ (b'_l) & \\ n+2\delta & (c_n),\ \Delta(\delta, \tfrac{1}{2}v \pm \tfrac{1}{2}\xi+1) & y \\ p & (d_p);\ (d'_p) & \end{array}\right] \cos \xi\theta\ ,$$

where δ is a positive integer, $u=0, 1, 2, \cdots; 0<\theta<\pi/2$. (8.5) and (8.6) are valid if $m+l<(n+p+1)$, $(m+l=n+p+1$; then $|x|, |y|<1)$, Re $(v)>0$, or if $m+l+1>n+p$, then $|\arg(y)|, |\arg(x)|<(m+l-n-p)\pi/2$.

9. Fourier series for generalized hypergeometric functions

In (8.5) and (8.6), taking $m=n$, $l=1$, $p=0$, and $y=x$, F-functions are reduced to the generalized hypergeometric functions of single variable. Hence we obtain the Fourier-cosine-series for generalized hypergeometric functions

$$(9.1) \qquad (\sin\theta)^v {}_{p+1}F_p\left\{\begin{matrix} a_1, \cdots, a_p, b+b' \\ c_1, \cdots, c_p \end{matrix}; x(\sin\theta)^{2\delta}\right\}$$

$$= \frac{2\prod_{i=0}^{\delta-1}\Gamma\left(\dfrac{v+1+2i}{2\delta}\right)\prod_{i=0}^{\delta-1}\Gamma\left(\dfrac{v+2+2i}{2\delta}\right)}{(\pi)^{3/2}\sqrt{(\delta)}} \sum_{\xi=0}^{\infty} \frac{\Gamma(\frac{1}{2}\pm\xi)}{\prod_{i=0}^{\delta-1}\Gamma\left(\dfrac{v\pm 2\xi+2+2i}{2\delta}\right)}$$

$$\times {}_{p+2\delta+1}F_{p+2\delta}\left\{\begin{matrix} a_1, \cdots, a_p, \Delta\left(\delta, \dfrac{v+1}{2}\right), \Delta\left(\delta, \dfrac{v+2}{2}\right), b+b' \\ c_1, \cdots, c_p, \Delta(\delta, \frac{1}{2}v\pm\xi+1) \end{matrix}; x\right\} \cos 2\xi\theta\ ,$$

$$(9.2) \qquad (\cos\theta)^v {}_{p+1}F_p\left\{\begin{matrix} a_1, \cdots, a_p, b+b' \\ c_1, \cdots, c_p \end{matrix}; x(\cos\theta)^{2\delta}\right\}$$

$$= \frac{2\prod_{i=0}^{\delta-1}\Gamma\left(\dfrac{v+1+2i}{2\delta}\right)\prod_{i=0}^{\delta-1}\Gamma\left(\dfrac{v+2+2i}{2\delta}\right)}{\sqrt{\pi}\,\delta^{1/2}} \sum_{\xi=0}^{\infty} \frac{1}{\prod_{i=0}^{\delta-1}\Gamma\left(\dfrac{v\pm\xi+2+2i}{2\delta}\right)}$$

$$\times {}_{p+2\delta+1}F_{p+2\delta}\left\{\begin{matrix} a_1, \cdots, a_p, \Delta\left(\delta, \dfrac{v+1}{2}\right), \Delta\left(\delta, \dfrac{v+2}{2}\right), b+b' \\ c_1, \cdots, c_p, \Delta(\delta, \frac{1}{2}v\pm\frac{1}{2}\xi+1) \end{matrix}; x\right\} \cos \xi\theta\ ,$$

where δ is a positive integer, $u=0, 1, 2, \cdots; 0<\theta<\pi/2$ and Re $(v)>0$.

10. Double-integral-expansion-analogues for generalized Meijer functions

With the help of the preceding sections we establish here some double-integral-expansion-analogues involving generalized Meijer functions $S\left[x\begin{Bmatrix}\sin\theta \\ \cos\theta\end{Bmatrix}^{2\rho}, y\begin{Bmatrix}\sin\varphi \\ \cos\varphi\end{Bmatrix}^{2\sigma}\right]$:

(10.1) $$\int_0^{\pi/2}\int_0^{\pi/2} \cos 2u\theta \cos 2m\varphi\, (\sin\theta)^v (\sin\varphi)^n\, S\begin{bmatrix} x(\sin\theta)^{2\rho} \\ y(\sin\varphi)^{2\sigma}\end{bmatrix} d\theta d\varphi$$

$$= \frac{(-1)^{u+m}\pi}{4\sqrt{(\rho\sigma)}} S\left[\begin{matrix} \begin{bmatrix} p\ , & 0 \\ A-p, & B \end{bmatrix} \\ \begin{pmatrix} q+2\rho, & r \\ C-q\ , & D-r+2\rho \end{pmatrix} \\ \begin{pmatrix} k+2\sigma, & l \\ E-k, & F-l+2\sigma \end{pmatrix} \end{matrix}\middle| \begin{matrix} (a);\ (b) \\ \Delta(\rho,\ -\frac{1}{2}v+\frac{1}{2}),\ \Delta(\rho,\ -\frac{1}{2}v),\ (c); \\ (d),\ \Delta(\rho,\ -\frac{1}{2}v\pm u) \\ \Delta(\sigma,\ -\frac{1}{2}n+\frac{1}{2}),\ \Delta(\sigma,\ -\frac{1}{2}n),\ (e); \\ (f),\ \Delta(\sigma,\ -\frac{1}{2}n\pm m) \end{matrix}\middle| \begin{matrix} x \\ \\ y \end{matrix}\right].$$

(10.2) $$\int_0^{\pi/2}\int_0^{\pi/2} \cos u\theta \cos m\varphi\, (\cos\theta)^v (\cos\varphi)^n\, S\begin{bmatrix} x(\cos\theta)^{2\rho} \\ y(\cos\varphi)^{2\sigma}\end{bmatrix} d\theta d\varphi$$

$$= \frac{\pi}{4\sqrt{(\rho\sigma)}} S\left[\begin{matrix} \begin{bmatrix} p\ , & 0 \\ A-p, & B \end{bmatrix} \\ \begin{pmatrix} q+2\rho, & r \\ C-q\ , & D-r+2\rho \end{pmatrix} \\ \begin{pmatrix} k+2\sigma, & l \\ E-k, & F-l+2\sigma \end{pmatrix} \end{matrix}\middle| \begin{matrix} (a);\ (b) \\ \Delta(\rho,\ -\frac{1}{2}v+\frac{1}{2}),\ \Delta(\rho,\ -\frac{1}{2}v),\ (c); \\ (d),\ \Delta(\rho,\ -\frac{1}{2}v\pm\frac{1}{2}u) \\ \Delta(\sigma,\ -\frac{1}{2}n+\frac{1}{2}),\ \Delta(\sigma,\ -\frac{1}{2}n),\ (e); \\ (f),\ \Delta(\sigma,\ -\frac{1}{2}n\pm\frac{1}{2}m) \end{matrix}\middle| \begin{matrix} x \\ \\ y \end{matrix}\right].$$

(10.3) $$(\sin\theta)^v(\sin\varphi)^n S\begin{bmatrix} x(\sin\theta)^{2\rho} \\ y(\sin\varphi)^{2\sigma}\end{bmatrix} = \frac{4}{\pi\sqrt{(\rho\sigma)}}\sum_{\delta=0}^{\infty}\sum_{\xi=0}^{\infty}(-1)^{\delta+\xi}$$

$$\times S\left[\begin{matrix} \begin{bmatrix} p\ , & 0 \\ A-p, & B \end{bmatrix} \\ \begin{pmatrix} q+2\rho, & r \\ C-q\ , & D-r+2\rho \end{pmatrix} \\ \begin{pmatrix} k+2\sigma, & l \\ E-k, & F-l+2\sigma \end{pmatrix} \end{matrix}\middle| \begin{matrix} (a);\ (b) \\ \Delta(\rho,\ -\frac{1}{2}v+\frac{1}{2}),\ \Delta(\rho,\ -\frac{1}{2}v),\ (c); \\ (d),\ \Delta(\rho,\ -\frac{1}{2}v\pm\delta) \\ \Delta(\sigma,\ -\frac{1}{2}n+\frac{1}{2}),\ \Delta(\sigma,\ -\frac{1}{2}n),\ (e); \\ (f),\ \Delta(\sigma,\ -\frac{1}{2}n\pm\xi) \end{matrix}\middle| \begin{matrix} x \\ \\ y \end{matrix}\right] \cos 2\delta\theta \cos 2\xi\varphi\ .$$

(10.4) $$(\cos\theta)^v(\cos\varphi)^n S\begin{bmatrix} x(\cos\theta)^{2\rho} \\ y(\cos\varphi)^{2\sigma}\end{bmatrix} = \frac{4}{\pi\sqrt{(\rho\sigma)}}\sum_{\delta=0}^{\infty}\sum_{\xi=0}^{\infty}$$

$$S\left[\begin{matrix} \begin{bmatrix} p\ , & 0 \\ A-p, & B \end{bmatrix} \\ \begin{pmatrix} q+2\rho, & r \\ C-q\ , & D-r+2\rho \end{pmatrix} \\ \begin{pmatrix} k+2\sigma, & l \\ E-k, & F-l+2\sigma \end{pmatrix} \end{matrix}\middle| \begin{matrix} (a);\ (b) \\ \Delta(\rho,\ -\frac{1}{2}v+\frac{1}{2}),\ \Delta(\rho,\ -\frac{1}{2}v),\ (c); \\ (d),\ \Delta(\rho,\ -\frac{1}{2}v\pm\frac{1}{2}\delta) \\ \Delta(\sigma,\ -\frac{1}{2}n+\frac{1}{2}),\ \Delta(\sigma,\ -\frac{1}{2}n),\ (e); \\ (f),\ \Delta(\sigma,\ -\frac{1}{2}n\pm\frac{1}{2}\xi) \end{matrix}\middle| \begin{matrix} x \\ \\ y \end{matrix}\right] \cos\delta\theta \cos\xi\varphi\ .$$

These double-integral-expansion-analogues are similar to our previous results (7.5), (7.9), (7.1) and (7.2) respectively.

The aforesaid results are valid under the following sets of conditions:

$$\text{(i)}\quad \begin{cases} 2(p+q+r)>[A+B+C+D], \\ \qquad |\arg(x)|<[p+q+r-\frac{1}{2}(A+B+C+D)]\pi\ , \\ 2(p+k+l)>[A+B+E+F], \\ \qquad |\arg(y)|<[p+k+l-\frac{1}{2}(A+B+E+F)]\pi\ ; \\ \operatorname{Re}[v+2\rho d_{h_1}]>0\ \ (h_1=1,2,\cdots,r); \\ \operatorname{Re}[n+2\sigma f_{h_2}]>0\ \ (h_2=1,2,\cdots,l)\ , \end{cases}$$

or

$$\text{(ii)}\quad \begin{cases} A+C<B+D,\ A+E<B+F \text{ or else} \\ A+C=B+D,\ A+E=B+F \text{ with } |x|<1,\ |y|<1\ , \\ \operatorname{Re}[v+2\rho d_{h_1}]>0\ \ (h_1=1,2,\cdots,r); \\ \operatorname{Re}[n+2\sigma f_{h_2}]>0\ \ (h_2=1,2,\cdots,l)\ , \end{cases}$$

where ρ and σ are positive integers, $u=0, 1, 2, \cdots$; $m=0, 1, 2, \cdots$; $0<\theta<\pi/2$ and $0<\varphi<\pi/2$.

11. On applications of Mellin's and Laplace's inversion formulae associated with generalized Meijer functions of two variables

Infinite integrals involving generalized Meijer functions.

We evaluate the following integrals which are required in our present investigation:

First integral:

$$\text{(11.1)}\qquad \int_0^\infty u^{\lambda-1}e^{-\rho u}S\begin{bmatrix} xu^\xi \\ yu^\xi \end{bmatrix}du$$

$$=(2\pi)^{(1-\xi)/2}\frac{\xi^{\lambda-1/2}}{\rho^\lambda}S\left[\begin{array}{c|c|c} \begin{bmatrix} p+\xi, & 0 \\ A-p, & B \end{bmatrix} & \Delta(\xi,\lambda),(a);(b) & \dfrac{x\xi^\xi}{\rho^\xi} \\ \begin{pmatrix} q\ , & r \\ C-q, & D-r \end{pmatrix} & (c);(d) & \\ \begin{pmatrix} k\ , & l \\ E-k, & F-l \end{pmatrix} & (e);(f) & \dfrac{y\xi^\xi}{\rho^\xi} \end{array}\right],$$

Second integral:

$$\text{(11.2)}\qquad \int_0^\infty \frac{u^{\lambda-1}}{(1+u)^\rho}S\begin{bmatrix} x\left(\dfrac{u}{1+u}\right)^\xi \\ y\left(\dfrac{u}{1+u}\right)^\xi \end{bmatrix}du$$

$$=\frac{\Gamma(\rho-\lambda)}{\xi^{\rho-\lambda}}S\left[\begin{array}{c|c|c} \begin{bmatrix} p+\xi, & 0 \\ A-p, & B+\xi \end{bmatrix} & \Delta(\xi,\lambda),(a);\Delta(\xi,\rho),(b) & x \\ \begin{pmatrix} q\ , & r \\ C-q, & D-r \end{pmatrix} & (c);(d) & \\ \begin{pmatrix} k\ , & l \\ E-k, & F-l \end{pmatrix} & (e);(f) & y \end{array}\right],$$

where ξ is a positive integer. (11.1) and (11.2) are valid under the following conditions:

$$
\text{(i)}\quad \begin{cases} A+B+C+D<2(p+q+r), \\ \qquad\qquad |\arg(x)|<[p+q+r-\frac{1}{2}(A+B+C+D)]\pi\ , \\ A+B+E+F<2(p+k+l), \\ \qquad\qquad |\arg(y)|<[p+k+l-\frac{1}{2}(A+B+E+F)]\pi\ ; \\ \mathrm{Re}\,(\lambda+\xi d_{h_1}+\xi f_{h_2})>0\ \ (h_1=1, 2, \cdots, r;\ h_2=1, 2, \cdots, l)\ , \\ \mathrm{Re}\,(\rho)>0,\ \mathrm{Re}\,(\rho-\lambda)>0 \end{cases}
$$

or

$$
\text{(ii)}\quad \begin{cases} A+C<B+D,\ A+E<B+F \\ (A+C=B+D,\ A+E=B+F,\ \text{then } |x|,\ |y|<1) \\ \mathrm{Re}\,(\lambda+\xi d_{h_1}+\xi f_{h_2})>0\ \ (h_1=1, 2, \cdots, r;\ h_2=1, 2, \cdots, l)\ , \\ \mathrm{Re}\,(\rho)>0,\ \mathrm{Re}\,(\rho-\lambda)>0\ . \end{cases}
$$

Proofs. (a) To prove (11.1), we substitute the double integral (1.1) in place of $S\begin{bmatrix} xu^\xi \\ yu^\xi \end{bmatrix}$ appearing in the integrand of (11.1), interchange the order of integration (which is permitted), we obtain

$$
\frac{1}{(2\pi i)^2}\int_{L_1}\int_{L_2}\Phi(s+t)\psi(s, t)\left\{\int_0^\infty e^{-\rho u}u^{\lambda+\xi s+\xi t-1}du\right\}x^s y^t ds dt\ .
$$

Regarding the interchange of the order of integration, it is observed that the u-integral is absolutely convergent if $\mathrm{Re}\,(\lambda+\xi s+\xi t)>0$, $\mathrm{Re}\,(\rho)>0$, the double contour-integral converges absolutely under the conditions referred to earlier, and the convergence of the repeated integral follows from that of the integral in (11.1). Hence the interchange of the order of integration is justified.

Now evaluating the u-integral with the help of the known result [6, p. 12, (33)] and using Gauss's multiplication formula etc., and in accordance with the definition (1.1), we obtain the value of the integral (11.1).

(b) The integral (11.2) is similarly evaluated on applying the same procedure as above and using the known result [6, p. 9, (2)].

Inversion formulae involving generalized Meijer functions.

We represent here inversion formulae for generalized Meijer functions of two variables.

(i) On applying Laplace's inversion formula (1.13) to (11.1), we have

$$
(11.3)\qquad u^{\lambda-1}S\begin{bmatrix} xu^\xi \\ yu^\xi \end{bmatrix}=(2\pi)^{(1-\xi)/2}\xi^{\lambda-1/2}
$$

$$\times\frac{1}{(2\pi i)}\int_{\sigma-i\infty}^{\sigma+i\infty}\rho^{-\lambda}$$

$$\times S\left[\begin{matrix}\begin{bmatrix}p+\xi, & 0\\ A-p, & B\end{bmatrix}\\ \begin{pmatrix}q\ , & r\\ C-q, & D-r\end{pmatrix}\\ \begin{pmatrix}k\ , & l\\ E-k, & F-l\end{pmatrix}\end{matrix}\left|\begin{matrix}\Delta(\xi,\lambda),(a);(b)\\ \\ (c);(d)\\ \\ (e);(f)\end{matrix}\right|\begin{matrix}\dfrac{x\xi^{\xi}}{\rho^{\xi}}\\ \\ \dfrac{y\xi^{\xi}}{\rho^{\xi}}\end{matrix}\right]e^{\rho u}d\rho\ .$$

(ii) On using Mellin's inversion formula (1.11) in (11.2), we get

$$(11.4)\qquad \frac{1}{(1+u)^{\rho}}S\left[\begin{matrix}x\left(\dfrac{u}{1+u}\right)^{\xi}\\ y\left(\dfrac{u}{1+u}\right)^{\xi}\end{matrix}\right]=\frac{1}{(2\pi i)}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{\Gamma(\rho-\lambda)}{\xi^{\rho-\lambda}}$$

$$\times S\left[\begin{matrix}\begin{bmatrix}p+\xi, & 0\\ A-p, & B+\xi\end{bmatrix}\\ \begin{pmatrix}q\ , & r\\ C-q, & D-r\end{pmatrix}\\ \begin{pmatrix}k\ , & l\\ E-k, & F-l\end{pmatrix}\end{matrix}\left|\begin{matrix}\Delta(\xi,\lambda),(a);\Delta(\xi,\rho),(b)\\ \\ (c);(d)\\ \\ (e);(f)\end{matrix}\right|\begin{matrix}x\\ \\ y\end{matrix}\right]u^{-\lambda}d\lambda\ ,$$

where ξ is a positive integer. These inversion formulae are valid under the following conditions

$$(\mathrm{i})\quad\begin{cases}A+B+C+D<2(p+q+r),\\ \qquad\qquad |\arg(x)|<[p+q+r-\frac{1}{2}(A+B+C+D)]\pi\ ,\\ A+B+E+F<2(p+k+l),\\ \qquad\qquad |\arg(y)|<[p+k+l-\frac{1}{2}(A+B+E+F)]\pi\ ;\\ \mathrm{Re}\,(\lambda+\xi d_{h_1}+\xi f_{h_2})>0\ \ (h_1=1,2,\cdots,r;\ h_2=1,2,\cdots,l),\\ \mathrm{Re}(\rho)>0,\ \mathrm{Re}\,(\rho-\lambda)>0\end{cases}$$

or

$$(\mathrm{ii})\quad\begin{cases}A+C<B+D,\ A+E<B+F\\ \quad(A+C=B+D,\ A+E=B+F,\ \text{then}\ |x|,|y|<1)\ ,\\ \mathrm{Re}\,(\lambda+\xi d_{h_1}+\xi f_{h_2})>0\ \ (h_1=1,2,\cdots,r;\ h_2=1,2,\cdots,l)\ ,\\ \mathrm{Re}\,(\rho)>0,\ \mathrm{Re}\,(\rho-\lambda)>0\ .\end{cases}$$

12. Particular cases of (11.3) and (11.4)

(a) Taking $A=B=0$, we get

$$(12.1)\qquad u^{\lambda-1}G_{C,D}^{r,q}\left\{xu^{\xi}\,\middle|\,\begin{matrix}(c)\\(d)\end{matrix}\right\}G_{E,F}^{l,k}\left\{yu^{\xi}\,\middle|\,\begin{matrix}(e)\\(f)\end{matrix}\right\}$$

$$=(2\pi)^{(1-\xi)/2}\xi^{\lambda-1/2}\frac{1}{(2\pi i)}\int_{\sigma-i\infty}^{\sigma+i\infty}\rho^{-\lambda}S\left[\begin{array}{c|c|c}\begin{bmatrix}\xi & , & 0\\ 0 & , & 0\end{bmatrix} & \Delta(\xi,\lambda);\cdots & \dfrac{x\xi^{\xi}}{\rho^{\xi}}\\ \begin{pmatrix}q & , & r\\ C-q, & D-r\end{pmatrix} & (c);(d) & \\ \begin{pmatrix}k & , & l\\ E-k, & F-l\end{pmatrix} & (e);(f) & \dfrac{y\xi^{\xi}}{\rho^{\xi}}\end{array}\right]e^{\rho u}d\rho\ ,$$

(12.2)
$$\frac{1}{(1+u)^{\rho}}G_{C,D}^{r,q}\left\{x\Big(\frac{u}{1+u}\Big)^{\xi}\,\Bigg|\,\begin{matrix}(c)\\(d)\end{matrix}\right\}G_{E,F}^{l,k}\left\{y\Big(\frac{u}{1+u}\Big)^{\xi}\,\Bigg|\,\begin{matrix}(e)\\(f)\end{matrix}\right\}$$

$$=\frac{1}{(2\pi i)}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{\Gamma(\rho-\lambda)}{\xi^{\rho-\lambda}}S\left[\begin{array}{c|c|c}\begin{bmatrix}\xi & , & 0\\ 0 & , & \xi\end{bmatrix} & \Delta(\xi,\lambda);\Delta(\xi,\rho) & x\\ \begin{pmatrix}q & , & r\\ C-q, & D-r\end{pmatrix} & (c);(d) & \\ \begin{pmatrix}k & , & l\\ E-k, & F-l\end{pmatrix} & (e);(f) & y\end{array}\right]u^{-\lambda}d\lambda\ ,$$

where ξ is a positive integer and

$$\text{(i)}\quad\begin{cases}C+D<2(q+r),\ |\arg(x)|<[q+r-\frac{1}{2}(C+D)]\pi\ ,\\ E+F<2(k+l),\ |\arg(y)|<[k+l-\frac{1}{2}(E+F)]\pi\ ;\\ \mathrm{Re}\,(\lambda+\xi d_{h_1}+\xi f_{h_2})>0\ \ (h_1=1,2,\cdots,r;\ h_2=1,2,\cdots,l)\ ,\\ \mathrm{Re}\,(\rho-\lambda)>0,\ \mathrm{Re}\,(\rho)>0\ ,\end{cases}$$

or

$$\text{(ii)}\quad\begin{cases}C<D,\ E<F\ \ (\text{if } C=D,\ E=F,\ \text{then } |x|,\ |y|<1)\ ,\\ \mathrm{Re}\,(\lambda+\xi d_{h_1}+\xi f_{h_2})>0\ \ (h_1=1,2,\cdots,r;\ h_2=1,2,\cdots,l)\ ;\\ \mathrm{Re}\,(\rho)>0,\ \mathrm{Re}\,(\rho-\lambda)>0\ .\end{cases}$$

(b) Substituting $A=p$, $E=k$, $l=1$, $f_1=0$, replace $A+C$ by A, $B+D$ by B, $A+q$ by s together with appropriate changes in the parameters and then make $y\to 0$, we have

(12.3)
$$u^{\lambda-1}G_{A,B}^{r,s}\left(xu^{\xi}\,\Bigg|\,\begin{matrix}(a)\\(b)\end{matrix}\right)$$

$$=(2\pi)^{(1-\xi)/2}\xi^{\lambda-1/2}\frac{1}{(2\pi i)}\int_{\sigma-i\infty}^{\sigma+i\infty}\rho^{-\lambda}G_{A+\xi,B}^{r,s+\xi}\left(\frac{x\xi^{\xi}}{\rho^{\xi}}\,\Bigg|\,\begin{matrix}\nabla(\xi,\lambda),\ (a)\\ (b)\end{matrix}\right)e^{\rho u}d\rho\ ,$$

(12.4)
$$\frac{1}{(1+u)^{\rho}}G_{A,B}^{r,s}\left(x\Big\{\frac{u}{1+u}\Big\}^{\xi}\,\Bigg|\,\begin{matrix}(a)\\(b)\end{matrix}\right)$$

$$=\frac{1}{(2\pi i)}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{\Gamma(\rho-\lambda)}{\xi^{\rho-\lambda}}G_{A+\xi,B+\xi}^{r,s+\xi}\left(x\,\Bigg|\,\begin{matrix}\nabla(\xi,\lambda),\ (a)\\ (b),\ \nabla(\xi,\rho)\end{matrix}\right)u^{-\lambda}d\lambda\ ,$$

where ξ is a positive integer and

$$A+B<2(r+s),\ |\arg(x)|<[r+s-\tfrac{1}{2}(A+B)]\pi,$$
$$\mathrm{Re}\,(\lambda+\xi b_j)>0\ \ (j=1,2,\cdots,r),\ \mathrm{Re}\,(\rho)>0,\ \mathrm{Re}\,(\rho-\lambda)>0\ .$$

In case $A=B$, $|x|<1$ and $\mathrm{Re}(\rho)>0$, $\mathrm{Re}(\rho-\lambda)>0$.

Special cases involving Meijer's G-functions

(i) Taking $\xi=1$ in (12.3), we have a known result [16, p. 15, (4.2)]:

$$x^{l-1}G_{p,q}^{m,n}\left(zx\left|\begin{matrix}a_1,\cdots,a_p\\ b_1,\cdots,b_q\end{matrix}\right.\right)$$
$$=\frac{1}{(2\pi i)}\int_{c-i\infty}^{c+i\infty}k^{-l}G_{p+1,q}^{m,n+1}\left(\frac{z}{k}\left|\begin{matrix}-l+1,a_1,\cdots,a_p\\ b_1,\cdots,b_q\end{matrix}\right.\right)e^{xk}dk$$

where $p+q<2(m+n)$, $|\arg(z)|<(m+n-\frac{1}{2}p-\frac{1}{2}q)\pi$, $\mathrm{Re}(k)>0$, $\mathrm{Re}(l+b_j)>0$ $(j=1,2,\cdots,m)$.

(ii) Setting $\xi=1$ in (12.4), we obtain a known result [16, p. 15, (4.1)]:

$$\frac{1}{(1+x)^k}G_{p,q}^{m,n}\left(\frac{zx}{1+x}\left|\begin{matrix}a_1,\cdots,a_p\\ b_1,\cdots,b_q\end{matrix}\right.\right)$$
$$=\frac{1}{(2\pi i)}\int_{c-i\infty}^{c+i\infty}\Gamma(k-l)G_{p+1,q+1}^{m,n+1}\left(z\left|\begin{matrix}-l+1,a_1,\cdots,a_p\\ b_1,\cdots,b_q,-k+1\end{matrix}\right.\right)x^{-l}dl\ ,$$

where $p+q<2(m+n)$, $|\arg z|<(m+n-\frac{1}{2}p-\frac{1}{2}q)\pi$, $\mathrm{Re}(k-l)>0$, $\mathrm{Re}(l+b_j)>0$ $(j=1,2,\cdots,m)$.

(c) Setting $p=A=m$, $B=n$, $q=k=C=E=l$, $r=l=1$, $D=F=p+1$, $d_1=f_1=0$, replace b_j, $1-c_j$, $1-d_j$, $1-e_j$ and $1-f_j$ by c_j, b_j, d_j, b'_j and d'_j respectively, then the generalized Meijer function $S\begin{bmatrix}x\\ y\end{bmatrix}$ reduces to Kampé de Féreit's double-hypergeometric function. Therefore in view of (1.4), we have

$$u^{\lambda-1}F_{n,p}^{m,l}\left[\begin{matrix}|a|_m\colon |b,b'|_l\\ |c|_n\colon |d,d'|_p\end{matrix}\left|\ xu^\xi,yu^\xi\right.\right]=(2\pi)^{(1-\xi)/2}\xi^{\lambda-1/2}\prod_{i=0}^{\xi-1}\Gamma\left(\frac{\lambda+i}{\xi}\right)$$
$$\times\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\rho^{-\lambda}F_{n,p}^{m+\xi,l}\left[\begin{matrix}|a|_m,\ \Delta(\xi,\lambda)\colon |b,b'|_l\\ |c|_n\qquad\qquad\colon |d,d'|_p\end{matrix}\left|\ \frac{x\xi^\xi}{\rho^\xi},\frac{y\xi^\xi}{\rho^\xi}\right.\right]e^{\rho u}d\rho\ , \tag{12.5}$$

$$\frac{1}{(1+u)^\rho}F_{n,p}^{m,l}\left[\begin{matrix}|a|_m\colon |b,b'|_l\\ |c|_n\colon |d,d'|_p\end{matrix}\left|\ x\left(\frac{u}{1+u}\right)^\xi,y\left(\frac{u}{1+u}\right)^\xi\right.\right]$$
$$=\frac{1}{\prod_{i=0}^{\xi-1}\Gamma\left(\frac{\rho+i}{\xi}\right)}\frac{1}{(2\pi i)}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{\Gamma(\rho-\lambda)}{\xi^{\rho-\lambda}}\prod_{i=0}^{\xi-1}\Gamma\left(\frac{\lambda+i}{\xi}\right)$$
$$\times F_{n+\xi,p}^{m+\xi,l}\left[\begin{matrix}|a|_m,\ \Delta(\xi,\lambda)\colon |b,b'|_l\\ |c|_n,\ \Delta(\xi,\rho)\colon |d,d'|_p\end{matrix}\left|\ x,y\right.\right]u^{-\lambda}d\lambda\ , \tag{12.6}$$

where ξ is a positive integer. (12.5), (12.6) are valid if $(m+l)<(n+p+1)$, $(m+l=n+p+1$; then $|x|,|y|<1)$, $\mathrm{Re}(\lambda)>0$, $\mathrm{Re}(\rho)>0$, $\mathrm{Re}(\rho-\lambda)>0$, or if

$m+l+1>n+p$, then $|\arg(y)|$, $|\arg(x)|<(m+l-n-p)\pi/2$, $\mathrm{Re}(\lambda)>0$, $\mathrm{Re}(\rho)>0$, $\mathrm{Re}(\rho-\lambda)>0$.

13. Some results involving Appell's functions

By setting the parameters suitably in (12.5) and (12.6), the double hypergeometric functions F are reduced to Appell's functions F_1, F_2, F_3 and F_4 respectively. Some interesting results are mentioned below:

(i) By virtue of (1.5), we obtain

$$(13.1)\quad u^{\lambda-1}F_1[a: b, b': c: xu^{\xi}, yu^{\xi}]$$
$$=(2\pi)^{(1-\xi)/2}\xi^{\lambda-1/2}\prod_{i=0}^{\xi-1}\Gamma\Big(\frac{\lambda+i}{\xi}\Big)\frac{1}{(2\pi i)}\int_{\sigma-i\infty}^{\sigma+i\infty}\rho^{-\lambda}$$
$$\times F_{1,0}^{1+\xi,1}\left[\begin{matrix}\Delta(\xi,\lambda), a: b, b'\\ c: \cdots\end{matrix}\middle| \frac{x\xi^{\xi}}{\rho^{\xi}}, \frac{y\xi^{\xi}}{\rho^{\xi}}\right]e^{\rho u}d\rho ,$$

$$(13.2)\quad \frac{1}{(1+u)^{\rho}}F_1\left[a: b, b': c: x\Big\{\frac{u}{1+u}\Big\}^{\xi}, y\Big\{\frac{u}{1+u}\Big\}^{\xi}\right]$$
$$=\frac{1}{\prod_{i=0}^{\xi-1}\Gamma\Big(\frac{\rho+i}{\xi}\Big)}\frac{1}{(2\pi i)}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{\Gamma(\rho-\lambda)}{\xi^{\rho-\lambda}}\prod_{i=0}^{\xi-1}\Gamma\Big(\frac{\lambda+i}{\xi}\Big)$$
$$\times F_{1+\xi,0}^{1+\xi,1}\left[\begin{matrix}\Delta(\xi,\lambda), a: b, b'\\ \Delta(\xi,\rho), c: \cdots\end{matrix}\middle| x, y\right]u^{-\lambda}d\lambda ,$$

where ξ is a positive integer, $|x|$, $|y|<1$ or $|\arg(y)|$, $|\arg(x)|<\pi/2$, $\mathrm{Re}(\lambda)>0$, $\mathrm{Re}(\rho)>0$, $\mathrm{Re}(\rho-\lambda)>0$.

(ii) Using (1.6), we have

$$(13.3)\quad u^{\lambda-1}F_2[a: b, b': d, d': xu^{\xi}, yu^{\xi}]$$
$$=(2\pi)^{(1-\xi)/2}\xi^{\lambda-1/2}\prod_{i=0}^{\xi-1}\Gamma\Big(\frac{\lambda+i}{\xi}\Big)\frac{1}{(2\pi i)}\int_{\sigma-i\infty}^{\sigma+i\infty}\rho^{-\lambda}$$
$$\times F_{0,1}^{1+\xi,1}\left[\begin{matrix}\Delta(\xi,\lambda), a: b, b'\\ \cdots\ : d, d'\end{matrix}\middle| \frac{x\xi^{\xi}}{\rho^{\xi}}, \frac{y\xi^{\xi}}{\rho^{\xi}}\right]e^{\rho u}d\rho ,$$

$$(13.4)\quad \frac{1}{(1+u)^{\rho}}F_2\left[a: b, b': d, d': x\Big\{\frac{u}{1+u}\Big\}^{\xi}, y\Big\{\frac{u}{1+u}\Big\}^{\xi}\right]$$
$$=\frac{1}{\prod_{i=0}^{\xi-1}\Gamma\Big(\frac{\rho+i}{\xi}\Big)}\frac{1}{(2\pi i)}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{\Gamma(\rho-\lambda)}{\xi^{\rho-\lambda}}\prod_{i=0}^{\xi-1}\Gamma\Big(\frac{\lambda+i}{\xi}\Big)$$
$$\times F_{\xi,1}^{1+\xi,1}\left[\begin{matrix}\Delta(\xi,\lambda), a: b, b'\\ \Delta(\xi,\rho)\ : d, d'\end{matrix}\middle| x, y\right]u^{-\lambda}d\lambda ,$$

where ξ is a positive integer, $|x|$, $|y|<1$ or $|\arg(y)|$, $|\arg(x)|<\pi/2$, $\mathrm{Re}(\rho)>0$, $\mathrm{Re}(\lambda)>0$, $\mathrm{Re}(\rho-\lambda)>0$.

(iii) In view of (1.7), we obtain

(13.5)
$$u^{\lambda-1}F_3[b_1, b_2; b_1', b_2': c: xu^{\xi}, yu^{\xi}]$$
$$=(2\pi)^{(1-\xi)/2}\xi^{\lambda-1/2}\prod_{i=0}^{\xi-1}\Gamma\left(\frac{\lambda+i}{\xi}\right)\frac{1}{(2\pi i)}\int_{\sigma-i\infty}^{\sigma+i\infty}\rho^{-\lambda}$$
$$\times F_{1,0}^{\xi,2}\left[\begin{matrix}\Delta(\xi, \lambda): b_1, b_2; b_1', b_2' \\ c \quad : \quad \cdots\end{matrix}\middle| \frac{x\xi^{\xi}}{\rho^{\xi}}, \frac{y\xi^{\xi}}{\rho^{\xi}}\right]e^{\rho u}d\rho ,$$

(13.6)
$$\frac{1}{(1+u)^{\rho}}F_3\left[b_1, b_2; b_1', b_2': c: x\left\{\frac{u}{1+u}\right\}^{\xi}, y\left\{\frac{u}{1+u}\right\}^{\xi}\right]$$
$$=\frac{1}{\prod_{i=0}^{\xi-1}\Gamma\left(\frac{\rho+i}{\xi}\right)}\frac{1}{(2\pi i)}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{\Gamma(\rho-\lambda)}{\xi^{\rho-\lambda}}\prod_{i=0}^{\xi-1}\Gamma\left(\frac{\lambda+i}{\xi}\right)$$
$$\times F_{1+\xi,0}^{\xi,2}\left[\begin{matrix}\Delta(\xi, \lambda): b_1, b_2; b_1', b_2' \\ \Delta(\xi, \rho), c: \quad \cdots\end{matrix}\middle| x, y\right]u^{-\lambda}d\lambda ,$$

where ξ is a positive integer, $|x|, |y|<1$, $|\arg(y)|, |\arg(x)|<\pi/2$, $\mathrm{Re}(\lambda)>0$, $\mathrm{Re}(\rho)>0$, $\mathrm{Re}(\rho-\lambda)>0$.

(iv) With the help of (1.8), we get

(13.7)
$$u^{\lambda-1}F_4[a_1, a_2: d, d': xu^{\xi}, yu^{\xi}]$$
$$=(2\pi)^{(1-\xi)/2}\xi^{\lambda-1/2}\prod_{i=0}^{\xi-1}\Gamma\left(\frac{\lambda+i}{\xi}\right)\frac{1}{(2\pi i)}\int_{\sigma-i\infty}^{\sigma+i\infty}\rho^{-\lambda}$$
$$\times F_{0,1}^{\xi+2,0}\left[\begin{matrix}\Delta(\xi, \lambda), a_1, a_2: \cdots \\ \cdots \quad : d, d'\end{matrix}\middle| \frac{x\xi^{\xi}}{\rho^{\xi}}, \frac{y\xi^{\xi}}{\rho^{\xi}}\right]e^{\rho u}d\rho ,$$

(13.8)
$$\frac{1}{(1+u)^{\rho}}F_4\left[a_1, a_2: d, d': x\left\{\frac{u}{1+u}\right\}^{\xi}, y\left\{\frac{u}{1+u}\right\}^{\xi}\right]$$
$$=\frac{1}{\prod_{i=0}^{\xi-1}\Gamma\left(\frac{\rho+i}{\xi}\right)}\frac{1}{(2\pi i)}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{\Gamma(\rho-\lambda)}{\xi^{\rho-\lambda}}\prod_{i=0}^{\xi-1}\Gamma\left(\frac{\lambda+i}{\xi}\right)$$
$$\times F_{\xi,1}^{\xi+2,0}\left[\begin{matrix}\Delta(\xi, \lambda), a_1, a_2: \cdots \\ \Delta(\xi, \rho) \quad : d, d'\end{matrix}\middle| x, y\right]u^{-\lambda}d\lambda ,$$

where ξ is a positive integer, $|x|, |y|<1$ or $|\arg(y)|, |\arg(x)|<\pi/2$, $\mathrm{Re}(\lambda)>0$, $\mathrm{Re}(\rho)>0$, $\mathrm{Re}(\rho-\lambda)>0$.

14. Inversion formulae involving generalized hypergeometric functions

In (12.5) and (12.6), setting $m=n$, $l=1$, $p=0$, $y=x$ and using (1.9), we obtain

(14.1)
$$u^{\lambda-1}{}_{p+1}F_p\left[\begin{matrix}a_1, \cdots, a_p, b+b' \\ c_1, \cdots, c_p\end{matrix}; xu^{\xi}\right]$$
$$=(2\pi)^{(1-\xi)/2}\xi^{\lambda-1/2}\prod_{i=0}^{\xi-1}\Gamma\left(\frac{\lambda+i}{\xi}\right)\frac{1}{(2\pi i)}\int_{\sigma-i\infty}^{\sigma+i\infty}\rho^{-\lambda}$$

$$\times {}_{p+\xi+1}F_p\left[\begin{matrix} a_1, \cdots, a_p, \Delta(\xi, \lambda), b+b' \\ c_1, \cdots, c_p \end{matrix}; \frac{x\xi^\xi}{\rho^\xi}\right] e^{\rho u} d\rho\,,$$

(14.2)
$$\frac{1}{(1+u)^\rho}\,{}_{p+1}F_p\left[\begin{matrix} a_1, \cdots, a_p, b+b' \\ c_1, \cdots, c_p \end{matrix}; x\left\{\frac{u}{1+u}\right\}^\xi\right]$$
$$=\frac{1}{\prod_{i=0}^{\xi-1}\Gamma\left(\frac{\rho+i}{\xi}\right)}\frac{1}{(2\pi i)}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{\Gamma(\rho-\lambda)}{\xi^{\rho-\lambda}}\prod_{i=0}^{\xi-1}\Gamma\left(\frac{\lambda+i}{\xi}\right)$$
$$\times {}_{p+\xi+1}F_{p+\xi}\left[\begin{matrix} a_1, \cdots, a_p, \Delta(\xi, \lambda), b+b' \\ c_1, \cdots, c_p, \Delta(\xi, \rho) \end{matrix}; x\right] u^{-\lambda} d\lambda\,,$$

where $\mathrm{Re}\,(\lambda)>0$, $\mathrm{Re}\,(\rho)>0$, $\mathrm{Re}\,(\rho-\lambda)>0$ and ξ is a positive integer.

15. Inversion formulae for generalized Sister Celin's polynomials

(i) In (14.1), substituting $\xi=\lambda=1$, $a_1=n+\alpha+\beta+1$, $c_1=1+\alpha$, $c_2=\frac{1}{2}$, $b+b'=-n$, etc., and multiplying both sides by $(1+\alpha)_n/n!$, we obtain a known result [18, p. 155, eqn. (4.4)]:

(15.1)
$$f_n^{(\alpha,\beta)}\left(\begin{matrix} a_2, \cdots, a_p \\ b_3, \cdots, b_q \end{matrix}; \mu x\right)=\frac{1}{(2\pi i)}\int_{\sigma-i\infty}^{\sigma+i\infty} s^{-1} f_n^{(\alpha,\beta)}\left(\begin{matrix} a_2, \cdots, a_p, 1 \\ b_3, \cdots, b_q \end{matrix}; \mu s^{-1}\right) e^{xs} ds\,,$$

where $\mathrm{Re}\,(s)>0$.

(ii) In (14.2), setting $\xi=1$, $b+b'=-n$, $a_1=n+\alpha+\beta+1$, $c_1=1+\alpha$, $c_2=\frac{1}{2}$, etc., and multiplying both sides by $(1+\alpha)_n/n!$, we get a known result [17, p. 20, (2.4)]:

(15.2)
$$\frac{\Gamma(m)}{(1+x)^m} f_n^{(\alpha,\beta)}\left(\begin{matrix} a_2, \cdots, a_p \\ b_3, \cdots, b_q \end{matrix}; \frac{\mu x}{1+x}\right)$$
$$=\frac{1}{(2\pi i)}\int_{\gamma-i\infty}^{\gamma+i\infty}\Gamma(l)\Gamma(m-l) f_n^{(\alpha,\beta)}\left(\begin{matrix} a_2, \cdots, a_p, l \\ b_3, \cdots, b_q, m \end{matrix}; \mu\right) x^{-l} dl\,,$$

where $\mathrm{Re}\,(l)>0$, $\mathrm{Re}\,(m-l)>0$.

We remark in passing that the generalized Meijer function $S\left[\begin{matrix} x \\ y \end{matrix}\right]$ not only includes the Meijer's G-functions and the product of two G-functions, as its particular cases on specializing the parameters, but it also provides most of the generally used functions in two arguments, e.g. Kampé de Fériet's double hypergeometric functions F [8], which in turn, yield the well-known Appell [1] functions F_1, F_2, F_3 and F_4, the Whittaker functions of two variables etc.

Acknowledgement

The author is highly indebted to the referee for some useful and valuable suggestions which led to a better presentation.

References

[1] APPELL, Paul and KAMPÉ DE FÉRIET, J.; *Fonctions hypergéométriques et hypersphériques; polynomes d'Hermites*, Gauthier-Villars, Paris, 1926.

[2] BROMWICH, T. J. I'A.; *Theory of infinite series*, Macmillan, London, 1926.

[3] BAJPAI, S. D.; *A Study of Meijer's G-function and MacRobert's E-functions and their applications*, Thesis approved for Ph. D. degree of University of Indore, Indore (M. P.), India, 1968.

[4] ERDÉLYI, A.; *Tables of Integral Transforms, I*, McGraw-Hill, New York, 1954.

[5] ERDÉLYI, A.; *Tables of Integral Transforms, II*, McGraw-Hill, New York, 1954.

[6] ERDÉLYI, A.; *Higher Transcendental Functions, I*, McGraw-Hill, New York, 1953.

[7] FASENMYER, Sister Mary Celine; Some generalized hypergeometric polynomials, *Bull. Amer. Math. Soc.* **53** (1947), 806–812.

[8] KAMPÉ de FÉRIET, J.; *La fonction hypergéométrique*, Gauthier-Villars, Paris, 1937.

[9] LUKE, Y. L.; *Integrals of Bessel-functions*, McGraw-Hill, New York, 1962.

[10] MCLACHLAN, N. W.; *Bessel-functions for Engineers*, Clarenden Press, Oxford, 1961.

[11] RAGAB, F. M.; Expansions of Kampé de Fériet's double hypergeometric functions of higher order, *J. Reine angew, Math.*, **212** (1963), 113–119.

[12] SNEDDON, Ian, N.; *Special functions of mathematical physics and chemistry*, Oliver and Boyd, Edinburgh and London, Interscience Publishers INC., New York, 1956.

[13] SHARMA, B. L.; On the generalized function of two variables, *Annales de Soc. Sci., de Bruxelles*, **79** (1965), 26–40.

[14] SHARMA, B.L.; *A generalized function of two variables*, Thesis approved for the Ph. D. degree of Jodhpur University, 1964.

[15] SHAH, Manilal.; Certain integrals involving the product of two generalized hypergeometric polynomials, *Proc. Nat. Acad. Sci.*, (India), Section A, **37** (1967), 79–96.[1]

[16] SHAH, Manilal; On applications of Mellin's and Laplace's inversion formulae to H-functions, *Labdev Journal of Science and Technology*, (India), **7-A** (1969), 10–17.[2]

[17] SHAH, Manilal; On applications of Mellin's inversion formula to hypergeometric polynomials, *Labdev Jour. of Science and Technology*, (India), **6-A** (1968), 19–22.[3]

[18] SHAH, Manilal; On applications of Mellin's and Laplace's inversion formulae to hypergeometric polynomials, *Vijnana Parishad Anusandhan Patrika*, Vijnana Parishad. Allahabad (India), **12** (1969), 151-156.

Department of Mathematics
P.M.B.G. Science College, Sanyogitaganj
Indore (M.P.), India

1) MR 39 #4454.
2) MR 39 #4457.
3) MR 40 #409.